27-301
Microstructure-Properties
Fracture: the Griffith Equation
Profs. A. D. Rollett, M. De Graef

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**Liberty ship, Comet heart valve failures**

This WW-2 era “Liberty” ship, with a welded steel hull, failed by brittle propagation of a crack while moored in dock! The fracture toughness of the steel was too low for the design.

e.g. Fundamentals of Fracture Mechanics by John Frederick Knott

The de Havilland Comet was one of the first jet-powered airplanes used in commercial service in the early 1950s. Although the designers had considered fracture, they did not allow for possible failure via fatigue from the pressurization-depressurization cycle of each flight.


This is an artificial heart valve. Failure has occurred in some types of valve, e.g. through breakage at the weld of the outlet strut to the carrier ring in the Björk-Shiley Convexo-Concave (BSCC) mechanical prosthetic heart valve.

Could It Happen Again? The Björk-Shiley Convexo-Concave Heart Valve Story, E Blackstone, Circulation, 2005;111:2717-2719

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Objective

• The objective of this lecture is to relate the fracture resistance of materials to their microstructure.
• Both ceramics and metals exhibit strongly microstructure dependent fracture resistance.
• This section focuses on basic theory of brittle fracture and the Griffith equation in particular:
  \[ \sigma_{\text{break}} = \frac{K_c}{\sqrt{\pi c}}. \]
• Part of the motivation for this lecture is to provide background for the 2nd Lab on the sensitivity of mechanical properties to microstructure. The fracture resistance of steels, for example, is very sensitive to the microstructure.
• The material property of interest is fracture toughness, which has dimensions of stress intensity and relates breaking strength to flaw/crack size.

†A. A. Griffith, *Phil. Trans. Soc. (Lond.) A* 221 (1920) 163.

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1. What is the Ductile-to-Brittle Transition Temperature? This is the temperature above which most crystalline materials become ductile.

2. What is stress intensity (K) and how is it used in the equation for brittle fracture? Stress intensity is a measure of the stress concentration at the tip of a crack and, once a critical value is reached, brittle fracture is expected.

3. What was Griffith’s interpretation of his experiments on glass fibers? He assumed that the diameter of the fiber controlled or limited the maximum flaw size and thus the breaking strength.

4. How do we derive the Griffith Eq.? By balancing the energy released during crack advance against the energy consumed in creating new surface. What roles do elastic energy and surface energy play? See above. How does plastic work modify this? Plastic work generally increases (drastically) the energy cost of advancing a crack.

5. How can we understand the Liberty ship problem? The cracking of Liberty ships arose from use of a steel whose DBTT was above the freezing point of water so that in cold waters the hull material was below the DBTT. Welded structures contributed to the problem.

6. What is the theoretical cohesive strength of a crystal (stress-based) and how do we derive it? We obtain the theoretical cohesive strength by assuming a simple elastic (non-linear) stress-strain curve and integrating to the point of maximum stress. How does this value relate to $E$? The initial slope equals Young’s modulus.

7. What is stress concentration? The tendency of stress to be locally higher than the far-field stress near a geometrically reduced cross-section. Formula for elliptical cracks? See the notes.

8. How do we derive a stress-based formula for the brittle strength? We obtain a stress-based equation by combining stress concentration at a crack tip with the theoretical cohesive strength (see above).

9. Compare and contrast the stress-based approach with the energy-based approach (Griffith) to fracture. The stress-based approach depends on the shape of the crack (e.g. the radius of curvature), which the energy-based Griffith Eq does not.

10. Why is it important to learn about the Griffith Eq.? The approach embodied in the Griffith Eq. has been very successful for engineering design over a long period.
References (books)

- J.E. Gordon (1978), Structures, or Why Things Don’t Fall Down, Plenum, New York ISBN 0306812835. This is a highly readable and educational book.

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Relevance

Fracture is more important than strength from an engineering point of view! When we build something, we want it to resist the loads that we put on it and not break. Of course, objects can come apart in a way that is not necessarily disastrous just through regular wear and tear. By fracture, we generally mean unanticipated (worse, unpredictable) breakage. Metallurgists are fond of quoting the Liberty ship experience because of its historical significance. Just consider the Firestone tire problem in the late 1990s, however, for a more contemporary example of the seriousness of fracture in a composite system (primarily rubber and steel). As an example of a polymer-ceramic composite, think of the inconvenience and pain of cracking or breaking a tooth (never mind a bone).

en.wikipedia.org/wiki/Firestone_and_Ford_tire_controversy
Definition of Fracture

- Fracture is the propagation of a crack across a loaded section.
- The material property that characterizes fracture resistance is its toughness or fracture toughness.
- Toughness is related to the energy per unit crack advance.
- From linear elastic fracture mechanics, the units of toughness are MPa√m. The reason for the odd-looking units is that the square root of the crack length or flaw size enters the (Griffith) equation.
- Just as for yield strength, toughness scales with elastic modulus.
- How do we relate a stimulus to a response, when the response is breakage and the stimulus is stress?! The answer is that, just like yield stress, fracture toughness is effectively a threshold property - below the critical stress intensity, nothing breaks but above it, fracture occurs (or is highly likely to occur). Thus fracture toughness is akin to the critical resolved shear stress - the material is elastic below it and plastic, flowing above it.
- Fracture toughness is a non-linear property because it is a threshold value, rather than a proportional response to a stimulus.
Fracture Characteristics

• The practical significance of fracture is sudden failure (breakage) at a load less than that which the part was designed to carry.

• Although shear stresses can generate fracture, 99% of practical examples involve tensile stresses.

• Tensile stresses are important because they can separate the atoms and lead to crack formation.

• Fracture can be either ductile or brittle.

• Brittle fracture implies little energy consumed in the process, e.g. breaking glass.

• Ductile fracture implies that much energy is expended in, e.g. breaking a pure metal.

• Ductility, or fracture toughness, is extremely important from an engineering and materials selection perspective, because it ensures that loads are re-distributed in a structure without failure. If the load exceeds the yield in some location, the material is more likely to flow plastically, thus causing some other part of the structure to absorb the load.
Concepts to be learned

- Cracks require energy to be supplied in order to create new surface area and propagate. We will see that the Griffith criterion is the result of considering an energy balance for each increment of crack advance.

- The energy consumed goes into - at a minimum - creating new surface area. We can measure the energy consumed as the area under the stress-strain curve - \( \int \sigma \, d\varepsilon \). The greatest amount of energy is consumed when plastic work must be performed in the course of crack propagation. Microstructure plays the largest role here (and where the materials engineer acts).

- The energy supplied comes from the elastic energy stored in a loaded body. As the crack propagates, some part of the body is unloaded, thereby releasing energy. Geometry plays the largest role here; e.g. stress concentration at the tip of a sharp crack (which is what a mechanical engineer must avoid!).

- In addition to the trade-off between energy supplied and consumed, there must be enough stress available to break bonds at the crack tip. Generally speaking, however, this criterion is satisfied and the energy criterion above is the governing one.
Fracture Depends on Microstructure

- The fracture properties (resistance to fracture) depend strongly on microstructure.
- If mechanisms exist for dissipating significant amounts of energy during crack propagation such as dislocation slip (plastic flow), then a material will have high toughness.
- Intrinsically ductile materials, such as most metals, have high toughness but must be strengthened to be made useful.
- Intrinsically brittle materials, such as most ceramics, have high strength but must be toughened to be made useful.
- Toughening mechanisms include plastic flow, grain size refinement, crack bridging, phase transformation, fiber reinforcement.
- Defects such as voids and poorly bonded second phase particles decrease toughness by introducing nucleation sites for cracks.
- Grain size refinement is often effective for increasing toughness because small cracks are limited to individual grains; the smaller the grain, the smaller the microcrack length and therefore the higher the stress required to propagate the crack.
- Residual stresses introduced by thermal cycling or deformation also tend to decrease toughness because they mean that there must be a region with elevated tensile stress (balanced by compressive stresses elsewhere).
Design

The importance of fracture mechanics in design of structures is paramount. The example of the Liberty ship in the 1940’s is classic: many ships were lost either at sea or even in dock because of the propagation of a crack around the welded steel hull. It turned out that failure could be associated with a Charpy test energy of less than 15 ft lbs at the temperature at which the accident happened. Thus was the link between toughness and design born. The specific point is that engineering design requires the designer to take account of all possible modes of fracture.
Charpy Impact testing is a method of assessing the fracture resistance of materials at moderately high loading rates.

The Charpy test is the most widely used evaluation technique for measuring the toughness of materials; it utilizes impact loading conditions. Standard-sized specimens (10 mm sq x 60 mm long) containing a sharp notch (2 mm deep with a .015 mm radius) to localize the stress, are hammer-impacted and the energy absorbed during this fracture process is measured. As the pendulum hammer has a fixed weight and drops the same distance each time (see Figure) its kinetic energy when it strikes the specimen is always the same. Part of this energy is consumed in breaking the specimen; the energy remaining in the hammer causes the pendulum to continue its upward swing. By measuring the difference in the height of the upward swing after the pendulum has fallen freely and after it has broken the sample, the energy absorbed in breaking the sample may be calculated. This energy is the impact strength of the material and can be read directly from a dial gauge on the machine.

The units of the result of a Charpy test are those of energy (e.g. Joules).
Practical Impact

• Despite the crude nature of the Charpy test, it is of great practical value. By “crude” we mean that there is no theory that allows one to calculate anything quantitative about crack propagation based on the impact energy. However, the variations in impact energy correlate very well with the more scientifically based fracture toughness test that provides a critical stress intensity value, $K_{IC}$. The critical stress intensity does provide a means of calculating the likelihood of brittle (i.e. sudden, catastrophic) crack propagation.

• We learn about fracture toughness by investigating the stress at which brittle cracks propagate in a material. This was the subject of a justly famous experiment by Griffith.

• Despite the success of this experiment in demonstrating the importance of brittle crack propagation, and the subsequent work by Inglis and others to link this concept to stress concentration, it was not until the Liberty Ship and Comet disasters that fracture toughness was applied to large structures.

• Even now, there is an astonishing lack of materials-based standards in vehicles. It appears that there is no standard for fracture toughness in ship hull steels, for example: see [http://www.safeship.ca/page7.php](http://www.safeship.ca/page7.php) [accessed 8 xi 15].
Griffith & glass fibres

• A.A. Griffith is considered to have made the first substantial scientific contribution to the understanding of brittle fracture (1920). He measured the breaking strength of glass fibres of varying thicknesses and found that their strength varied in inverse proportion to their diameter, see fig. from Green’s book (next slide). He then showed that Inglis’s equation for the stress [concentration] at the root of an elliptical crack could be applied to the problem to rationalize the results. That is, by assuming that the largest flaw was of order of the fibre diameter, he could demonstrate that his results were consistent with Inglis’s theory.

• Note that glass (especially the types that contain sodium) develop cracks over time through interaction with water vapor in the atmosphere. Freshly prepared glass fibers, or ones that have been flame-annealed, are much stronger than those that have set around (in air) for a period of time.


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Figure 8.2 from Green’s book on Ceramic Mechanical Behavior, for strength of glass fibers, as determined by Griffith. Note the inverse relationship between size and strength.

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Microstructure Effect on Fracture

• Whether or not a material fractures on loading depends on a *competition between flow and fracture*. If flow [of dislocations] is easy then fracture will only occur when necking (localization) happens. If flow is difficult then fracture will relieve the loading instead.

• Microstructure: weakly bonded second phase particles tend to promote fracture by acting as initiation sites for cracks.

• Fine grain size tends to inhibit fracture by providing a high density of crack arrest/deflection points. Also, even if a grain cracks, then the stress concentration at the end of the crack decreases with decreasing crack size [= grain size].
Temperature Effect

• Temperature: temperature affects plasticity in many materials. Higher temperatures promote deformation whereas low temperatures promote fracture. In many materials, a ductile-to-brittle transition can be detected as you lower the temperature, see fig 11.8 from Cottrell (next slide). The reason for this is rising yield stresses with decreasing T eventually mean that tensile stresses cause fracture before shear stress can cause deformation.

• This also illustrates the essential aspect of competition between fracture and plastic flow. If dislocation slip is easy, then even a artificially made crack will blunt by plastic flow at its tip.

• Why do we care (as materials engineers)? Think about ships, especially those that sail in northern climates in water at freezing point. If the toughness of the hull steel drops to low values around freezing point then a crack may be able to propagate suddenly and catastrophically!
Ductile-Brittle Transition Temperature (DBTT)

This is known as the Ludwik-Davidenkov criterion

[Cottrell]

breaking stress (brittle)

yield stress (ductile)

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Ductile-Brittle Transition Temperature (DBTT)

- The basic explanation of the transition from ductile to brittle failure as the temperature is decreased is that, as the temperature is decreased, it becomes too difficult to move dislocations (as quantified by the critical resolved shear stress) relative to the stress required to propagate a crack (as quantified by the tensile breaking stress).

- Most materials exhibit a $\text{crss}$ that increases rapidly with decreasing temperature. This is generally caused by an increase in the Peierls stress, i.e. the lattice friction (after the British physicist, Rudolf Peierls).

- The *Peierls stress* denotes the stress required to move a dislocation from one atom position to the next (along the close-packed direction). In fcc metals, this stress is negligible. In bcc metals at low temperatures, however, this stress is non-trivial. In most ceramics, the Peierls stress is so high (e.g. silicon) at room temperature that dislocations cannot be moved. The Peierls stress is *independent* of other obstacles to dislocation motion and depends only the *atomic structure*.
Environment, Loading

• Environment: certain interstitial elements are particularly deleterious, e.g. hydrogen in steels. The exact causes are still subject to scientific debate but the presumption is that atomic hydrogen lowers the cohesive energy between the planes that tend to cleave in brittle fracture. Hydrogen can be readily introduced in welding, e.g. by use of damp weld rods. Ammonia is notorious as a promoter of corrosion fatigue, e.g. cracking in brass. Similarly chloride ions (salt) in iron alloys (even stainless steel!).

• Type of loading: multiaxial stresses involving tension promote fracture whereas stresses involving compression promote deformation, especially if deviatoric stresses are maximized. Monotonic loading is generally less severe than cyclic loading. Specimen design is also critical – notches promote fracture over deformation.
Brittle Fracture: overview

• What are we trying to establish about brittle fracture?
• Real materials always have cracks (or potential cracks): therefore one must always be concerned about whether the cracks will grow catastrophically.
• Defects with high aspect ratios (cracks!) lead to stress concentrations which help cracks to grow.
• The most useful idea (Griffith) is that of an energy balance between elastic energy release rate and surface energy consumption. In brittle solids, this energy barrier is easily overcome.
• We need to understand these basic concepts in order to comprehend the role of microstructure!
Pulling atoms apart – theoretical cohesive strength

- We will return to the basic concept of needing enough stress at the tip of a crack in order to break bonds and propagate the crack.
- For the moment, however, ignore the issue of stress concentration in order to estimate the maximum possible tensile breaking stress.
- Remember that this is a necessary but not sufficient condition.
- Step 1 = calculate the theoretical cohesive strength
- Step 2 = calculate the effect of a crack
- Finish = relate fracture strength to modulus, crack geometry

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Theoretical cohesive strength: stress based

- The theoretical cohesive strength of a solid is very high - between 1/4 and 1/50 of its Young’s modulus.

- Approximate the interatomic force-displacement or stress-strain curve with a sine law with initial slope = E, and equate the result to the macroscopic elastic relationship:

  \[ \sigma = \sigma_c \sin\left(\frac{\pi x}{(\lambda/2)}\right). \]

- We need a way to estimate the amplitude, \( \sigma_c \), which is the theoretical fracture strength that we want.

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Theoretical cohesive strength: stress based

- Sine Curve: \( \sigma = \sigma_c \text{sin}\{\pi x/(\lambda/2)\} = \sigma_c \text{sin}\{2\pi x/\lambda\} \)

Small displacements: linearize: \( \sigma = \sigma_c 2\pi x/\lambda \)

The slope near the origin: \( d\sigma/dx = \sigma/x = \sigma_c 2\pi/\lambda \)

Compare with Hooke’s Law:

\[
\text{strain} = \Delta l/l = x/a_0 \\
E = \text{stress/strain} = \sigma / (x/a_0) = \sigma a_0/x \\
\therefore \quad E/a_0 = \sigma/x
\]

where \( a_0 \) is the equilibrium distance between atoms.

From equating the two expressions for the slope:

\[
E/a_0 = \sigma/x = \sigma_c 2\pi/\lambda \\
E/a_0 = \sigma_c 2\pi/\lambda
\]

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Theoretical cohesive strength: stress based

• Re-arranging to obtain the breaking stress, we obtain:

\[ \sigma_c = \frac{E\lambda}{2\pi a_0} \]

or,

\[ 2\sigma_c a_0 / E = \frac{\lambda}{\pi} \]

• If we approximate the spacing (wavelength) for the sine curve as the interatomic spacing, so that \( \lambda \approx a_0 \), then

\[ \sigma_c = \frac{E}{2\pi} \]

which means that one estimate of the theoretical strength is about 1/7 (simplified to 1/10 by some authors) of the Young’s modulus.

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**Theoretical cohesive strength: energy based**

- An alternative approach is to consider the energy balance inherent in this problem: we can estimate the work done to separate the two surfaces from these equations, and equate it to the surface energies. Work to fracture the solid, based on the sine curve:

\[
\text{work done} = \int \sigma d\varepsilon = \int_0^{\lambda/2} \sigma_c \sin \left( \frac{\pi x}{\lambda/2} \right) dx = \sigma_c \frac{\lambda}{\pi} = 2\gamma_{\text{surface}}
\]

- By substituting the previously obtained relationship for \( l \) (to avoid making any assumptions about it) into the above relationship:

\[
2\sigma_c a_0 / E = \frac{\lambda}{\pi}, \quad \text{which gives: } 2(\sigma_c)^2 a_0 / E = 2\gamma,
\]

we can obtain the following estimate of the fracture strength in terms of modulus, surface energy and bond length:

\[
\sigma_{\text{break}} = \sqrt{\frac{E\gamma}{a_0}}
\]

- NO PRE-EXISTING CRACKS assumed for this derivation. The resulting estimate gives similar values to the first one.
Estimates of Fracture Strength

- Evidently, the two different estimates of fracture strength give similar results.
- On this basis there is nothing to choose between the stress-based and the energy-based criteria.

<table>
<thead>
<tr>
<th>Material</th>
<th>E/10 (GPa)</th>
<th>$\sqrt{E\gamma/a_0}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au</td>
<td>7.8</td>
<td>17.7</td>
</tr>
<tr>
<td>Cu</td>
<td>12.1</td>
<td>24.8</td>
</tr>
<tr>
<td>Ni</td>
<td>22.5</td>
<td>37.2</td>
</tr>
<tr>
<td>NaCl</td>
<td>3.7</td>
<td>6.3</td>
</tr>
<tr>
<td>MgO</td>
<td>31</td>
<td>40.8</td>
</tr>
<tr>
<td>TiC</td>
<td>~45</td>
<td>28.5</td>
</tr>
<tr>
<td>Si</td>
<td>16</td>
<td>28.5</td>
</tr>
</tbody>
</table>

- Based on the observed maximum strengths of whiskers and fibers, the ratio of modulus:strength approaches that of the theoretical estimates.
- The last two materials are technological materials with useful, albeit small ductility.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\sigma_{\text{break}}$ (GPa)</th>
<th>E (GPa)</th>
<th>$E/\sigma_{\text{break}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silica fiber</td>
<td>24.1</td>
<td>97.1</td>
<td>4</td>
</tr>
<tr>
<td>Iron whisker</td>
<td>13.1</td>
<td>295</td>
<td>23</td>
</tr>
<tr>
<td>Silicon whisker</td>
<td>6.5</td>
<td>166</td>
<td>26</td>
</tr>
<tr>
<td>Alumina whisker</td>
<td>15.2</td>
<td>496</td>
<td>33</td>
</tr>
<tr>
<td>Ausformed steel</td>
<td>3.14</td>
<td>200</td>
<td>64</td>
</tr>
<tr>
<td>Piano wire</td>
<td>2.75</td>
<td>200</td>
<td>73</td>
</tr>
</tbody>
</table>

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Stress Concentration

Stress concentration is an extremely important concept. The local increase in stress around a void in a material concentrates the stress next to it, far above the level that you would compute from the change in cross-sectional area.

- Stress concentration at the corners of the windows of the Comet airplane contributed to its fatigue problem.
- Note the pattern of fringes in the upper right figure, shows the concentration of stress around the tip of the crack.
- Polarized light is shone through the specimen. Polycarbonate rotates the plane of polarization of light (birefringence), so using crossed polars reveals contours of maximum shear stress.

Figure 2.19 Photoclastic fringes in polycarbonate
[Materials in Principle and Practice].

Figure 2.20

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**Stress Concentration**

- Now we return to the basic concept of stress concentration at the tip of a crack that is higher than the average stress in the body. We can then ask what combination of stress and crack geometry gives a high enough stress to break bonds and propagate the crack.

- We employ a formula by Inglis for an elliptical crack of length $c$ and thickness $b$:

  \[
  \frac{\sigma_{\text{maximum}}}{\sigma_{\text{applied}}} = 1 + \frac{2c}{b}
  \]

- The sharper the crack, the greater the stress concentration at the crack tip, $\sigma_{\text{maximum}}$.  

[FIGURE 7.5 Elliptical hole in infinitely large panel produces stress concentration of $1 + 2a/b$.]

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Fracture limited by stress concentration

- Now notice that the radius of curvature, $\rho$, at the crack tip (sharp end of the ellipse) is given by:

$$\rho = \frac{b^2}{c}$$

- Combining the previous equation with this one and then simplifying $1 + 2\sqrt{c/\rho}$ to $2\sqrt{c/\rho}$ because $c \gg \rho$,

$$\sigma_{\text{maximum}} = 2\sigma_{\text{applied}}\sqrt{c/\rho}$$

- The factor $2\sqrt{c/\rho}$ is sometimes referred to as a stress concentration factor, $k$.

- The net result is that, with a crack present, less external stress is needed in order to reach the breaking stress of the material.

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**Stress concentration + Breaking stress**

- What if we now assume that brittle fracture occurs when the stress at the crack tip reaches the theoretical maximum tensile stress at which atomic bonds are ruptured (based on our second, energy-based criterion)?

- Set the maximum local stress as follows:

  \[ \sigma_{\text{maximum}} = \sigma_{\text{break}} = \sqrt{\frac{E\gamma}{a_0}}. \]

- Combine with the Inglis equation and the net result is:

  \[ \sigma_{\text{break}} = \sqrt{\frac{\gamma E \rho}{4a_0c}}. \]

- This tells us that the breaking strength depends on both crack size, \( c \), and the sharpness of the crack, \( \rho \).

- Contrast this with the previous result for theoretical cohesive strength that ignores the effect of stress concentration:

  \[ \sigma_{\text{break}} = \sqrt{\frac{E\gamma}{a_0}}. \]

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Elastic Energy vs. Crack Surface (Griffith)

• Now we turn our attention to the role of energy, as opposed to just stress concentration. The previous derivation looked for a local stress-based criterion for crack propagation. This turns out to be a necessary but not sufficient condition for a crack to grow. We will now look for an energy-based criterion, as did Griffith.

• As a crack (dark green) advances, so a region of radius ~ crack length unloads (light orange). The elastic energy of this region is now available to do work, i.e. create new surface area.

• The elastic energy of the body goes down as the square of the crack length whereas the energy consumed in creating new surface area in a longer crack is proportional (linear) to the crack length. Note that the relevant length, \( c \), is the half-length of an elliptical crack.
Energy Balance: 2

- In order to compute the energy balance as the crack lengthens we need,
  
  (a) the elastic energy released = $\pi \sigma^2 c^2 t/E$
  and (b) the energy consumed = $4 ct\gamma$.

- To understand the balance point, consider the energy of the system, $U_{TOT}$, as a function of crack length (similar to the nucleation criterion):

$$U_{TOT} = -\pi \sigma^2 c^2 t/E + 4ct\gamma$$

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Griffith equation

• To find the “peak of the curve”, apply standard calculus and find the zero of the derivative.

\[
\frac{d}{dc} \left[ -\frac{\pi \sigma^2 c^2}{E} + 4c \gamma \right] = 0; \quad \sigma_{\text{break}} = \sqrt{\frac{2\gamma E}{\pi c}}
\]

• Notice that the “break-even point” for getting more elastic energy back than you have to put into the crack surface is stress dependent: higher stress, shorter crack required. Also the failure stress is not dependent on the crack shape (i.e. sharpness of the tip).

• Contrast this with the result for theoretical cohesive strength that ignores the effect of stress concentration:

\[\sigma_{\text{break}} = \sqrt{E\gamma/a_0},\]

and the result that includes stress concentration:

\[\sigma_{\text{break}} = \sqrt{E\gamma \rho/4a_0c}.\]
Stress Intensity, $K$

- Practical experience tells us that the Griffith approach is surprisingly general. Materials show characteristic stress intensities at which cracks will propagate. The presumption is that the former, stress-based criterion, is generally satisfied before (i.e. at lower stresses) the energy criterion is.
- Therefore we define a stress intensity as $K = \sigma \sqrt{c}$.
- Cracking is defined by $K > K_c$, where $K_c$ is a critical stress intensity or fracture toughness.

$$\sigma_{\text{break}} = K_c / \sqrt{\pi c}$$

- Can also define a toughness, $G_c$, which is given by

$$\sigma_{\text{break}} = \sqrt{(EG_c / \pi c)}$$

and allows us to modify (increase) the apparent surface energy to account for plastic work at the crack tip. It does, however, mean that there is no longer a direct link to the surface energy.
- The presence of $\sqrt{c}$ in the definition of stress intensity might look a bit odd because the other formulae have $\sqrt{\pi c}$ in them. However, consider multiplying both sides of the last equation above by $\sqrt{c}$ and what you get is this:

$$\sigma_{\text{break}} \sqrt{c} = \sqrt{(EG_c / \pi c)} \sqrt{c} \iff \sigma \sqrt{c} = \sqrt{(EG_c / \pi)} = \text{constant}.$$ 

In other words, the product of stress and square root of crack length is predicted (by Griffith) to be constant.
Plane Stress vs. Plane Strain

- *Plane stress* is associated with thin sheets, such that there are only two non-zero components of (normal) stress and the component through the thickness is zero.

- *Plane strain* is associated with thick sheets or plates, such that the component of through-thickness strain is zero.

- In both cases, the word “plane” implies a state of stress or strain that is confined to a plane.

- The previous derivation was for *plane stress*.

- By repeating the derivation for *plane strain*, one obtains a lower critical stress.

- Although this suggests that the breaking stress is larger for the plane strain case, it is generally the case that plane strain is more conservative than plane stress.

\[
\begin{pmatrix}
\sigma_{11} & 0 & 0 \\
0 & 0 & \sigma_{33} \\
0 & 0 & \epsilon_{33}
\end{pmatrix}
\]

\[
\sigma_{\text{break}} = \sqrt{\frac{2E\gamma}{\pi c}}
\]

\[
\sigma_{\text{break}} = \sqrt{\frac{2E\gamma}{\pi (1 - \nu^2)c}}
\]
# Fracture Toughness Values

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s Modulus (GPa)</th>
<th>Toughness (kJ/m²)</th>
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<tbody>
<tr>
<td>Tool steel</td>
<td>210</td>
<td>10-110</td>
</tr>
<tr>
<td>Mild steel</td>
<td>210</td>
<td>100</td>
</tr>
<tr>
<td>Ti-6Al-4V</td>
<td>120</td>
<td>20-110</td>
</tr>
<tr>
<td>Al 2014</td>
<td>70</td>
<td>7-30</td>
</tr>
<tr>
<td>Wood, across grain</td>
<td>0.2-0.7</td>
<td>8-20</td>
</tr>
<tr>
<td>Glass fiber composite</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>thermoplastics</td>
<td>1-2.5</td>
<td>2-8</td>
</tr>
<tr>
<td>Wood, along grain</td>
<td>6-16</td>
<td>0.5-2</td>
</tr>
<tr>
<td>Glassy thermoplastics (PMMA)</td>
<td>3</td>
<td>0.3-0.4</td>
</tr>
<tr>
<td>Epoxy (thermoset)</td>
<td>4</td>
<td>0.1-0.3</td>
</tr>
<tr>
<td>Alumina</td>
<td>390</td>
<td>0.06</td>
</tr>
<tr>
<td>Concrete</td>
<td>16</td>
<td>0.03</td>
</tr>
<tr>
<td>Window glass</td>
<td>72</td>
<td>0.01</td>
</tr>
</tbody>
</table>

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Summary

• Fundamental equations governing *brittle fracture* have been reviewed.

• Griffith demonstrated as a balance between the rate of [elastic] energy release versus energy consumed in creating new crack surface area.

• This approach leads to the concept of a critical *stress intensity*, written as $K_{IC}$, where the “I” in the subscript “IC” means that the material is loaded in *plane strain*. Plane strain (zero strain along one principal axis) gives the lowest measured toughness and is therefore the most conservative estimate of toughness.

• Ductile-to-Brittle Temperature (DBTT) introduced as characteristic of most materials. Related to the relative ease of plastic deformation versus brittle crack propagation.
Supplemental Slides

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Basic concept no. 1: crack tip stress

• We return to the basic concept of how to estimate the stress required at the tip of a crack in order to break bonds and propagate the crack.

• As before, we ignore the issue of stress concentration in order to estimate the maximum possible tensile breaking stress.

• The new aspect of this derivation is that we use the bulk modulus instead of the Youngs modulus. Ultimately this is motivated by the correlation that is known between toughness and the ratio of the shear modulus to the bulk modulus (which translates to the Poisson’s ratio).

• Remember that satisfying the minimum stress criterion this is a necessary but not sufficient condition for fracture to occur.

• Step 1 = calculate the theoretical cohesive strength  
  Step 2 = calculate the effect of a crack  
  Finish = relate fracture strength to bulk modulus

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**Bulk Modulus vs. Cohesive Strength**

- Why use the bulk modulus instead of Young’s modulus? The answer is that bulk or, better, hydrostatic components of stress and strain are more closely related to the cohesive energy.
- Consider the classical picture of a potential energy as a function of atomic separation, from which the bulk modulus can be derived, amongst many other aspects of solid state behavior.
- As atoms are pulled further apart than their equilibrium spacing, so their potential energy is raised. At large enough spacings the (restoring) force that brings them back together decreases increasingly rapidly.
- The key assumption here is that it is reasonable to think in terms of the average spacing of the atoms (rather than the separation distances in specific directions).
- Recall the standard formula that connects bulk modulus to Young’s modulus and the Poisson’s ratio:
  \[ K = \frac{E}{3(1-2\nu)} \]
- Also the equivalent for the shear modulus:
  \[ G = \frac{E}{2(1+\nu)} \]
- From which we obtain the ratio of the shear:bulk moduli:
  \[ \frac{G}{K} = 1.5 \frac{(1-2\nu)}{(1+\nu)} \]

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Hydrostatic Strains

- For any type of stress and strain, it is simple to calculate the “bulk” or hydrostatic components. The most straightforward approach is to derive them from the (second rank) tensor quantities:

  \[ \sigma_{\text{hydrostatic}} = \frac{1}{3} \sigma_{ii} \]
  \[ \varepsilon_{\text{hydrostatic}} = \frac{1}{3} \varepsilon_{ii} \]

- The connection between bulk modulus and Poisson’s ratio is illustrated with a tensile stress thus:
  - for the case of zero ratio, the atoms move apart in response to the tensile stress and no lateral strain develops - bulk strain develops rapidly;
  - for the case of Poisson’s ratio = 0.5, the atoms move further apart in the tensile direction but come together in the lateral direction to compensate (exactly) for the tensile stretch - no bulk strain develops.

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Theoretical cohesive strength

• The theoretical cohesive strength of a solid is very high - between 1/4 and 1/50 of its bulk modulus.

• Approximate the interatomic force-displacement curve with a sine law and equate the result to the macroscopic elastic relationship:

\[ \sigma = \sigma_c \sin\left\{\frac{\pi x}{\lambda/2}\right\}. \]
Theoretical cohesive strength: 2

- Sine Curve: $\sigma = \sigma_c \sin\{\pi x/(\lambda/2)\} = \sigma_c \sin\{2\pi x/\lambda\}$

Small displacements: linearize: $\sigma = \sigma_c 2\pi x/\lambda$

The slope near the origin: $d\sigma/dx = \sigma_c 2\pi x/\lambda$

Compare with the relationship for the bulk modulus:

$K = (\text{bulk stress})/(\text{bulk strain}) = \sigma/(x/a_0)$

where $a_0$ is the equilibrium distance between atoms, such that $x/a_0$ is the strain. From this we get

$$d\sigma/dx = K/a_0$$

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Theoretical cohesive strength: 3

- Combine the two expressions for the slope of the stress-strain relationship:
  \[ \sigma_c = \frac{K\lambda}{2\pi a_0} \]

- If we approximate the spacing for the sine curve as the interatomic spacing, so that \( \lambda \approx a_0 \), then
  \[ \sigma_c = \frac{K}{2\pi} \]

- Finally, consider the energy balance inherent in this problem: we can estimate the work done to separate the two surfaces from these equations, and equate it to the surface energies.
Theoretical cohesive strength: 4

• Work to fracture the solid, based on the sine curve:

\[
\text{work done} = \int_0^{\lambda/2} \sigma_c \sin \frac{\pi x}{\lambda/2} \, dx = \sigma_c \frac{\lambda}{\pi} = 2\gamma_{\text{surface}}
\]

• By substituting this into the previous relationship, we can obtain the following estimate of the fracture strength in terms of modulus, surface energy and bond length:

\[
\sigma_{\text{break}} = \sqrt{K\gamma/a_0}
\]

• NO PRE-EXISTING CRACKS assumed for this derivation.

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Correlation of Toughness with Modulus Ratio

• It is interesting to speculate that the correlation of the toughness of materials with the ratio of the shear modulus to the bulk modulus has to do with the competition between nucleating and propagating dislocations (in the crack tip), dependent on the shear modulus, versus breaking atoms apart to propagate a brittle crack, dependent on the bulk modulus.

• As the $G/K$ ratio decreases, so the Poisson’s ratio increases. Lower shear modulus compared to the bulk modulus means larger shear strains and higher probability of nucleating a dislocation (and propagating it) compared to pulling atoms far enough apart to overcome the cohesive energy and propagate a brittle crack.

• This section was developed as a follow-up to a discussion with Prof. Lindsay Greer, Materials & Metallurgy, Cambridge University.
Elastic Strain Energy Release Rate

- Another approach to this issue consists in considering the elastic strain energy density, or, more usefully, the energy release rate with crack advance (which is what enters the Griffith Eq.).

\[
G = \frac{1 - \nu^2}{E} \left( K_I^2 + K_{II}^2 \right) + \frac{1 + \nu}{E} K_{III}^2
\]

- Note that the Stress Intensity factors, \( K \), depend only on the far field applied stress and the crack geometry. Note that the relatively simple formula given above only applies to plane strain loading. This is a reasonable case to examine, however, because it is generally considered to correspond to the most conservative estimate of fracture toughness.

- Consider what we see if we assume that only mode-I loading is present, i.e. only \( K_I \) is non-zero. By definition, the stress intensities are given by the following, where \( s_{22} \) is the far field applied stress.

\[
K_I = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{22} \quad K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{12} \quad K_{III} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{32}
\]

- This suggests that, for constant Young’s modulus the elastic energy release rate goes down as Poisson’s ratio increases (which means that increasing Poisson’s ratio corresponds to increasing bulk modulus, holding \( E \) constant).

Web pages written by A.F. Bower on Solid Mechanics: sections 9.3, 9.4

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G and Fracture Toughness

• The simple minded approach to understanding the relationship between toughness and Poisson’s ratio is as follows. Assume that we hold Young’s modulus and the surface energy constant, where the surface energy is a notional quantity that sets the critical energy release rate, $G_C$. Then consider the standard criterion that brittle fracture occurs when the following is satisfied:

$$G \geq G_C,$$

so,

$$G = \frac{1 - \nu^2}{E} K_I^2 = \frac{1 - \nu^2}{E} 2\pi r \sigma_{22}^2 \geq G_C$$

• This shows that increasing Poisson’s ratio results in needing a larger applied stress in order to reach the required critical energy release rate.