Lecture Objectives: Composites

- The main objective of this lecture is to introduce you to microstructure-property relationships in composite materials.
- Composite materials constitute a huge class of materials. The objective of this lecture will therefore be to provide some definitions and describe some of the basic relationships.
- Cellular materials will be emphasized because of their connection to natural materials (biomaterials) and especially wood, which some of you will study in the second Lab.
Questions & Answers for Part 1

1. What are the general advantages of composite materials over monolithic materials? Give both biomaterial and man-made examples. Composites generally have higher specific properties. Wood and carbon-fiber reinforced plastics are examples.

2. What is the rule of mixtures as applied to composites? Integrate the property of interest over the volume of the composite.

3. What do the terms isostrain and iso-stress mean? As implied, iso-stress means same stress in all materials; iso-strain means same strain in all materials. For iso-stress you can think of the phases as being connected in series between the planes across which the load is transmitted (and vice versa for iso-strain).

4. Derive the isostrain model. See the notes; derivation relies on averaging the strains in the different phases.

5. Derive the iso-stress model. See the notes; derivation relies on averaging the stresses in the different phases.

6. Sketch the variations in modulus expected for composites in which the components have strongly different moduli. See the notes; iso-strain model gives linear variation (same as Rule of Mixtures in this case) whereas iso-stress model gives non-linear variation.

7. Explain what is meant by the Voigt, Reuss and Hill average moduli. Voigt= iso-strain, Reuss= iso-stress, Hill averages these two.

8. Which model for stiffness applies to a composite material with a compliant matrix and a well dispersed particulate second phase that is stiffer (than the matrix)? In this case, the Reuss (iso-stress) model applies because the individual particles are not connected and thus there is little load transfer between them.

9. Which model for stiffness applies to a composite material with a compliant matrix and a well dispersed, parallel, stiff fibers that is loaded along the fiber direction? In this case, the Voigt (iso-strain) model applies because the individual fibers are strained equally with the matrix.

10. Why are cellular or foam materials useful for achieving low modulus? By making a substantial fraction of the "material" empty space (air or trapped gas), one can reduce the modulus to the volume average of the solid material and gas. This accesses modulus values that are inaccessible to fully dense materials.

Key points

- Composites are regarded as artificial (man-made) mixtures of phases.
- Classification of composites by reinforcement type (dimensionality) - particles, fibers and laminated.
- Application of the Rule of Mixtures.
- Dependence of composite properties on the spatial arrangement of the phases.
- Upper and lower bounds on properties - example of elastic modulus, Voigt and Reuss approximations.
- High property:density ratios achievable with composites.
- Engineering with residual stress in composites.
- Anisotropy of composite properties, e.g. elastic modulus.
- Properties of wood as a cellular material.
- Cellular/foam materials as shock absorbers.
Examples

What are Composites?

- Composite materials contain more than one phase.
- Almost all materials contain more than one phase, so what’s the difference?
- The term *composite* is typically applied to a material when the multi-phase structure is constructed by direct intervention (external to the material).
- Composite Material Examples: glass fiber reinforced plastic (GRP), wood, clam shell, Mars bar.
- Multi-phase Material Examples: precipitation strengthened aluminum alloys, Ti-6Al-4V, dual-phase steel, transformation toughened alumina (Al$_2$O$_3$-CeO$_2$).
- Caution! There is some overlap between the categories!
Properties

- It is useful to review the basic properties of the different types of materials that are used in composites.
- Polymers - long [carbon] chain molecules with anything from van der Waals bonding between the chains (thermoplastics) to covalent links (thermosets). Low density, low modulus compared to other materials. Often highly formable (ductile).
- Ceramics - ionic or covalent bonding, lower symmetry crystal structures, high melting point and modulus, resistant to degradation, brittle, high modulus.
- Metals - metallic bonding, symmetric crystal structures, medium melting point, medium modulus, ductile, formable, variable resistance to degradation.

Why Use Composites? [Biomaterials]

- In nature, the basic materials tend to be weak and/or brittle. Evolution has resulted in structures that combine materials together for properties that far exceed those that could be obtained in the basic materials.
- The basic inorganic constituent of bone, for example, is calcium phosphate in the form of crystalline Ca_{10}(PO_4)_{6}(OH) and amorphous CaPO_3. This ceramic is brittle and not particularly stiff. The matrix of fibrous collagen is tough but even less stiff. When embedded arranged in the form of a cellular material, however, remarkable values of stiffness:density and toughness:density are achieved (and land-based multi-tonne creatures are possible such as elephants).
- A similar situation exists in wood where the basic materials are quite compliant but arranged in the multi-level composite forms that we know, high values of strength:density and toughness:density result.
Why Use Composites? [Man-made]

- The basic reason for the use of composites is always the same: some combination of properties can be achieved that is impossible in a monolithic material [for a given cost].
- In SiC-reinforced aluminum for brake rotors, for example, the combination of light weight, toughness (from the Al matrix at \(\sim 2.7 \text{ Mgm/m}^3\)), and stiffness (from the SiC additions) is not possible in either constituent by itself.
- In Cu-Nb for high strength electrical conductors, the combination of >1 GPa yield strength and high electrical conductivity (in the Cu) could never be achieved in either constituent by itself. In this case the high strength is a synergistic property of the composite.

Key aspects of composites

- Composites are expensive to make, as compared to monolithic materials, especially if the shape and arrangement of the phases must be controlled.
- Therefore there must be a strong motivation for making a composite structure to offset the cost.
- The simplest composites are particulate composites. Laminates are next, followed by fiber composites. Woven structures are the most complex.
Typical Microstructures

- We show next some typical microstructures.
- In biomaterials, many are cellular composites at some length scale (typically around 1 μm).
- Man-made composites are more often fully dense. The three major [structural] material types are all used so the abbreviations MMC [metal matrix composite], CMC [ceramic matrix composite], and PMC [polymer matrix composite] are commonly used.

Cellular Biomaterials

Note the variation in density; also the presence of distinct layers of cells in some woods, and in bone.

Note also that the shape of the cells and their walls makes a difference to their properties.

Gibson & Ashby: Cellular Solids
Man-made Examples

Note the typical length scale of ~100µm, and the use of fibers for reinforcement. This basic type of fiber-reinforced composite is strongly anisotropic. The toughness of such composites and the need for limited adhesion between fiber and matrix is discussed in the lecture on Fracture.

Dowling: Mech. Behavior Materials

Food!

From left to right, top to bottom:

a) Bread
b) Meringue
c) Chocolate bar
d) Chip
e) Malteser (Candy)
f) Jaffa cake (cookie, see below)

Gibson & Ashby: Cellular Solids
Food for Thought!

- How does ice cream represent a material in which the thermal-mechanical history is critical to its microstructure which, in turn, controls its properties?
- Hint: this involves both the properties of composite materials (ice, cream, voids) and particle coarsening (the ice).

Examples of composites

- The classical example of a composite is concrete.
- It is more complex than it appears. There are typically coarse and fine particles (rocks!) embedded in a matrix of silicates and sulfates. There is a high fraction of pores of all sizes. This is an example of a particulate composite.
- Ordinary concrete (properly made) has excellent compressive strength but poor tensile strength. Thus reinforced concrete was invented to combine the tensile strength of steel with the compressive strength of concrete. This is an example of a multiscale particulate and fiber reinforced composite. It is particulate because the aggregate (coarse gravel) reinforces the cement, and fiber because the steel rods reinforce the concrete.
- A subtle but very important variant of reinforced concrete is pre-stressed concrete in which the reinforcing rods are placed in tension before the concrete is allowed to set. See following slides on residual stresses.
Glass-ceramics

- Glass ceramics are useful materials that combine chemical inertness with thermal stability. They typically are stronger than amorphous glasses.
- This material class was invented (by the Sandia National Laboratories) for the specific purpose of making a material (insulator) that would have a good match for the thermal expansion characteristics of metals (stainless steel, nickel alloys), i.e. a relatively high CTE with values intermediate between ceramics (typically low) and metals (typically high).
- Typical phase mixture includes lithium silicate(s), cristobalite and residual glass phase.

Property Ranges

A much wider range of properties is possible in composites than in monolithic materials. Foams permit much smaller moduli and densities than fully dense materials. The following chart illustrates a few basic properties.

Gibson & Ashby: Cellular Solids
Notation

\[ A, B, C \] phases A and B, Composite
\[ V_A \] volume fraction of phase A
\[ P_A \] property of phase A
\[ E_A \] (Young’s) modulus of phase A
\[ \varepsilon_C \] (average) strain in composite
\[ \sigma_A \] stress in phase A
\[ K \] bulk modulus
\[ G \] shear modulus
\[ \alpha \] coefficient of thermal expansion (CTE)
\[ \rho_w \] cell wall density
\[ \rho^* \] relative density (cellular material)
\[ l \] length (of a beam)
\[ t \] thickness
\[ b \] depth
\[ \delta \] displacement
\[ K_{IC} \] fracture toughness [plane strain]
\[ P \] load
\[ x \] position (or location, in a material)

Simple Models: Rule of Mixtures

- What is the simplest model that can be used to predict a material property in a composite?
  Answer: Rule of Mixtures

- Define the volume fractions, \( V \), of the various materials comprising the composite. The average property of the composite is then given by,

  \[
  P_C = V_A P_A + V_B P_B + \ldots = \sum V_i P_i
  \]

  \[
  P_C = \int P(x) dV
  \]

- The Rule of Mixtures is an acceptable first approximation for estimating composite material properties. It is, however, often considerably in error and better methods are required.
**Limits, bounds**

- There are some circumstances under which one would like to be able to make quantitative predictions of the properties of a composite but an exact solution is not available.
- Under these circumstances, it is still possible to set limits on the property. In a formal sense these limits are known as *bounds* because they are the result of analysis using the principles of solid mechanics. Such analysis can demonstrate that an either an upper or a lower (or both) bound exists for a given structure and loading.
- An upper bound means that the value of the property cannot go any higher than a certain value and vice versa for a lower bound.

**Exact versus bounds**

- Exact solutions are usually available for simple geometries. Reinforced concrete with parallel rods is such an example.
- Complex geometries are almost always limited to approximate solutions and bounds provide the best estimate. Most particulate composites, especially those with cracks fall in this category.
- The most interesting properties for this lecture are those associated with mechanical behavior such as stiffness, strength, thermal expansion.
- In the most general sense, we are seeking methods for averaging a property over the heterogeneous elements of the microstructure.
**Isostress, isostrain**

- “Iso-” is a prefix meaning “same”. Isostress is an assumption that the phases experience the same stress. By contrast, isostrain makes the assumption that the phases are subject to the same strain.
- Each assumption leads to very different results, especially when the properties of each phase are divergent, as we see from the example of the brick and the foam.

**Isostrain**

- Imagine parallel slabs of material between platens that apply a load.
- Phase A has volume fraction $V_A$ and modulus $E_A$; Phase B has volume fraction $V_B$ and modulus $E_B$.
- Composite modulus, $E_C$?
- We assume *isostrain* because each phase sees the same change in length.
- The strain, $\varepsilon = \varepsilon_C$, is therefore the field; the stress is the response (and the stiffness is the property).
**Isostrain: 2**

- Each phase gives a different stress: \( \sigma_A = E_A \varepsilon_C; \ \sigma_B = E_B \varepsilon_C. \)
- We average the stresses over the composite in proportion to the volume fraction of the phase:
  \[ \sigma_C = V_A \sigma_A + V_B \sigma_B = V_A E_A \varepsilon_C + V_B E_B \varepsilon_C. \]
- The modulus is the ratio of the stress to the strain in the composite:
  \[ E_C = \sigma_C / \varepsilon_C = V_A E_A + V_B E_B. \]
- This modulus is thus the *arithmetic mean* of the moduli of each phase, weighted by the volume fractions. In effect, the *rule of mixtures* has been applied to the stiffnesses.
- Exercise: prove to yourself that this can be extended to any number of phases.

**Isostress**

- Imagine parallel slabs of material between platens that apply a load.
- Phase A has volume fraction \( V_A \) and modulus \( E_A \); Phase B has volume fraction \( V_B \) and modulus \( E_B \).
- Composite modulus, \( E_C \)?
- We assume *isostress* because each phase sees the *same stress* (assuming same cross-sectional area).
- The stress, \( \sigma = \sigma_C \), is therefore the field; the strain is the response (and the compliance is the property).
Isostress: 2

- Each phase gives a different strain:
  \[ \varepsilon_A = \frac{\sigma_C}{E_A}; \varepsilon_B = \frac{\sigma_C}{E_B}. \]
- We average the strains over the composite in proportion to the volume fraction of the phase:
  \[ \varepsilon_C = V_A \varepsilon_A + V_B \varepsilon_B = \frac{V_A \sigma_C}{E_A} + \frac{V_B \sigma_C}{E_B}. \]
- The modulus of the composite is the ratio of the stress to the strain in the composite as before, except that it is easier to work with inverse moduli, i.e. compliances:
  \[ \frac{1}{E_C} = \frac{\varepsilon_C}{\sigma_C} = \frac{V_A}{E_A} + \frac{V_B}{E_B}. \]
- The composite modulus is thus the harmonic mean of the moduli of each phase, weighted by the volume fractions. In effect, the rule of mixtures has been applied to the compliances.

Example of Cu-W Composites

- The graphs show (a) examples of the difference in the calculated modulus based on the 2 different assumptions [parallel is equivalent to isostrain, and series to isostress]; (b) an example of the measured difference in modulus of Cu-W composites, contrasting wire (=fiber) with particle reinforcement. The fiber composite corresponds very closely to the isostrain estimate; the particulate composite is close to the isostress, although not quite so precisely.
- Note how the isostress and isostrain estimates are similar when the moduli differ by only a factor of two. When the moduli differ by an order of magnitude, however, the two estimates differ widely.
- Here, isostrain happens to be the same as the Rule of Mixtures.
Voigt, Reuss, Hill

- These simple estimates of modulus have names associated with them.
- The *isostress* approach is known as the Reuss modulus.
- The *isostrain* approach is known as the Voigt modulus.
- Hill proposed that a reasonable average of the two would be appropriate in materials where the loading is intermediate between the two extreme cases. Hence the average of the Isotress and Isostrain values is known as the Hill Average Modulus.
- We can also treat the composite property (for elastic modulus) in terms of an *arithmetic mean* (isostrain) versus a *harmonic mean* (isostress), which is the reciprocal of the average of the reciprocal values.
- Are there better estimates? Yes, look in the Supplemental slides for a description of the Hashin-Shtrikman estimates of modulus. There are also textbooks by Mura, Nemat-Nasser, Milton and several others.

Homework Questions

- No worked example is provided here on the iso-strain and iso-stress models.
- Examples were quoted of theoretical combinations of materials and for Cu-W.
- Homework/exam questions are likely to ask you to calculate modulus values at different volume fractions (of two phases), to plot the results (linear or log scale) and to compare against experimental data.
- You may be asked to rationalize deviations of measured modulus values from calculated ones by considering microstructure. For example, if a particulate composite (with stiff particles in a compliant matrix, e.g. SiC in Al) has higher modulus than you compute from the iso-stress model, then this may be because the particles are not perfectly dispersed and they form networks through inter-particle contacts.
**Summary: Part 1**

- Composites are man-made mixtures of phases, often with different material types, e.g. glass (ceramic) as a stiffening reinforcement in epoxy (polymer).
- The simplest way to estimate properties is to use the *Rule of Mixtures*. Such simple volume averaging is also valid for field quantities such as stress or strain, depending on boundary conditions.
- The next simplest approach to computing the properties of a composite is to look for upper and lower bounds. For the example of elastic modulus, the iso-strain and iso-stress models were developed. The iso-strain model happens to give the same result as the Rule of Mixtures but has a physical basis.

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**Part 2**

- In this Part, we consider the properties of wood.
- Wood is a multiscale composite material, in the sense that it is self-evidently a cellular material but the cell walls are themselves composite structures.
- Wood is a natural example of a cellular material.
- We also examine the anisotropy of composite materials, partly as a way of tying together what we learned about anisotropy with what we learn about composite structures.
Questions & Answers for Part 2

1. What makes wood a multiscale composite material? Wood is a multiscale composite material because there is identifiable structure at the scale of filaments, microfibrils, cell wall layers and cell organization (“grain”).

2. What are the main chemical components of wood? Wood contains mostly cellulose (in various forms) and lignin.

3. What distinguishes wood from other plants? The main difference between wood and other plants is that its cell walls contain lignin, which makes it stronger and more resistant to pests.

4. What is the macrostructure of wood? Wood contains highly elongated cells, that define the “grain” of wood. Cells are deposited on a nearly continuous basis but their diameter varies during the year with larger diameter cells during times of rapid growth. There are also radial structures known as “rays”.

5. What is the structure of the cell walls? There are several layers. There is a Primary outer layer (P), outside of which there is a “middle lamella” that contains most of the lignin. Inside the P layer, there are the S1, S2 & S3 layers, with different layups of the microfibrils.

6. What are “rings” in wood? As noted above, the cell size varies on an annual basis which means that a cross-section through a trunk reveals what look like rings in the structure; each ring corresponds to one year, which permits the age of a tree to be estimated with good reliability. The variations in cell size also reveal changes in local climate.

7. What microstructural characteristic correlates most strongly with the mechanical properties of wood? Properties such as modulus, strength and fracture toughness correlate most strongly with density.

8. How does elastic modulus depend on density in wood? The elastic modulus varies either in proportion to density for the axial/longitudinal direction or with the density squared across the grain (radial or circumferential).

9. Explain the structure and components of micro-fibrils. Each microfibril is a bundle of cellulose fibers in a matrix of hemicellulose and lignin.

10. Explain the structure of cell walls in wood. See the notes; Several layers are present in the cell wall, each with its own characteristic lay-up angle of the micro-fibrils. In particular, the angle between the microfibrils and the axial direction in the S2 layer is strongly anti-correlated with stiffness.

11. Describe the anisotropy of the mechanical properties of wood. Wood is much stiffer, stronger and tough parallel to the grain than across the grain.

12. Based on the chemistry of wood, comment on its sensitivity to moisture. Certain components of the wood (esp. cellulose) are hydrophilic and absorb water. Increased moisture content increases stiffness and strength.

13. Why does the modulus vary faster than linear across the grain? Crucially, wood is a cellular material and deforms primarily via bending of the cell walls.

Wood: Macro-structure

Note the variation in cell size during the year from Earlywood (spring-summer) to Latewood (summer-autumn). This variation in cell size produces the characteristic “rings” that indicate the age of the wood because of the yearly cycle in cell size (and the magnitudes of the cell sizes correlate with climatic conditions). The “Rays” are aligned with the radial direction. The long direction of the cells is the axial or longitudinal direction. Follow a ring around the trunk and this is the circumferential or tangential direction.
It is important to understand wood as a cellular, composite structure. It is one, however, that has several different length scales from that of the cellulose molecule to the macrostructure of lumber as we accustomed to looking at it at the visual scale. The figure illustrates the hierarchy of length scales from the atomic structure of cellulose (A) to the structure of a tree trunk (E). The basic building block of wood is the polymer of glucose known as cellulose, which occurs as a (mostly) crystalline fiber. The other critical component of wood is lignin, which is a complex, amorphous material containing phenyl groups. Lignin sets wood apart from other plants; its occurrence in the outer and inner linings of the cell walls is critical for both structural properties and for wood’s (relative) insensitivity to environment.
**Wood: cell structure**

- Each cell wall contains microfibrils, each of which is a bundle of cellulose fibers in a matrix of hemicellulose and lignin. Therefore it is a *fiber-reinforced composite!* The *P* layer is 5% of the thickness with random fiber directions; the *S*₁ layer is 9% with fibers at 50-70° w.r.t. the axis; the *S*₂ layer is 85%, fibers at 10-30°; the *S*₃ layer is 1%, with fibers at 90° to the axis. Note the dependence of the tensile strength on the microfibrill angle in the Outer Wall, labeled “S₁”. Each cell is a long tube, some of which are used for transporting water (but not all).

![Cell structure diagram]

**Wood: Microfibril structure**

- The filaments or fibers of cellulose, \((C_6H_{10}O_5)_n\), where \(n\sim 10^4\), are organized in bundles (together with lignin surrounding the fibers) called microfibrils whose size is about 10 nm. Each set of microfibrils forms a bundle that is itself a structural member of the wall of a cell (next slide).
- Soft woods have longer cellulose fibers than hardwoods (which matters to the manufacture of paper).
Wood: Constituents: Cellulose

- Cellulose: a high molecular weight, stereoregular, and linear polymer of repeating beta-D-glucopyranose units. It is the main structural element and major constituent of the cell wall of trees and plants. The empirical formula for cellulose is \( (C_6H_{10}O_5)_n \) where 'n' is degree of polymerization (DP).

[http://www.paperonweb.com/wood.htm]

<table>
<thead>
<tr>
<th>Substance</th>
<th>Degree of Polymerization (DP)</th>
<th>Molecular Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Native Cellulose</td>
<td>&gt;3500</td>
<td>&gt;570,000</td>
</tr>
<tr>
<td>Purified Cotton</td>
<td>1000 - 3000</td>
<td>150,000 - 500,000</td>
</tr>
<tr>
<td>Wood Pulp</td>
<td>600 - 1000</td>
<td>90,000 - 150,000</td>
</tr>
<tr>
<td>Commercial Regenerated Cellulose (e.g. Rayon)</td>
<td>200 - 600</td>
<td>30,000 - 150,000</td>
</tr>
<tr>
<td>Y Cellulose</td>
<td>15 - 90</td>
<td>3000 - 15,000</td>
</tr>
<tr>
<td>X Cellulose</td>
<td>&lt;15</td>
<td>&lt;3000</td>
</tr>
<tr>
<td>Dynamite Nitro-Cellulose</td>
<td>3000 - 5000</td>
<td>750,000 - 875,000</td>
</tr>
<tr>
<td>Plastic Nitro-Cellulose</td>
<td>500 - 600</td>
<td>125,000 - 150,000</td>
</tr>
<tr>
<td>Commercial Cellulose Acetate</td>
<td>175 - 360</td>
<td>45,000 - 100,000</td>
</tr>
</tbody>
</table>

Taking wood as an example, it is found empirically that the moduli vary with (relative) density anisotropically.

\[
E_{\text{axial}} \propto E_{\text{cell}} \frac{\rho}{\rho_{\text{cell}}} \\
E_{\text{transverse}} \propto E_{\text{cell}} \left( \frac{\rho}{\rho_{\text{cell}}} \right)^2
\]

Note the discrepancy between the empirical equation and the slope in the plot. The theoretical prediction goes as \((\rho/\rho_{\text{cell}})^3\).

[Fig. 10.12. The Young’s modulus of wood plotted against density. One pair of axes is normalized by the cell wall Young’s modulus in the axial direction (35 GPa) and by the cell wall density (1500 kg/m³). The other pair of axes corresponds to the raw data. Data: Goodman and Bodig (1970); Bodig and Goodman (1973); Wood Handbook (1974); Denwoodie (1981); Bodig and Jayne (1982); and Easterling et al. (1982).]

[Fig. 10.12. Young’s Modulus vs. Density.]

Cellular Materials: Young’s Modulus
Wood: Young’s Modulus

- To understand what controls the elastic modulus (Young’s modulus) of wood, we have to consider bending of the cell walls in the microstructure.

- The first equation quantifies the idea that the tensile modulus of wood parallel to the grain is just the volume average of the area fraction occupied by cell wall.

\[
E_{\text{axial}} \propto E_{\text{cell}} \frac{\rho}{\rho_{\text{cell}}} \tag{1}
\]

\[
E_{\text{transverse}} \propto E_{\text{cell}} \left( \frac{\rho}{\rho_{\text{cell}}} \right)^2 \tag{2}
\]

![Diagram of cell edge bending during linear-elastic deformation.]

Wood: Modulus, contd.

- The second equation (modulus transverse to the grain) is more subtle and states that the elastic modulus varies more rapidly - with the square of the density - than the axial modulus. The reason for this can be understood very simply in terms of the cellular structure. When wood is loaded across the grain, the cell walls bend like miniature beams. This response can be quantified by use of beam theory to arrive at the functional dependence of equation 2.
Summary: Part 2A

- Wood can be understood as a composite material or, more usefully, as a **cellular** material.
- Wood is a multi-scale composite material.
- The cell walls of wood are themselves composite structures.
- Even the fibers in the cell walls are also composites.
- The elastic properties of wood are highly anisotropic: wood is stiffer in the axial direction and more compliant in the transverse direction.
- The variation in modulus with relative density is linear in the axial direction but varies as the square of the relative density in the transverse direction.
- In part 2B, we introduce beam bending theory to quantify these effects.

2B: Introduction to beam theory

- Consider a 3-point beam with length, \( l \): supported at either end and loaded in the center with a force, \( F \). The most important point is that there is a neutral point in the beam, \( n \), at which the stress is zero; above this it is compressive, and below it is tensile. The stress is proportional to distance from the neutral plane.
Beam theory applied to wood

- The mechanical behavior can be modeled by a framework of beams. The deflection, $\delta$, of a beam of length $l$ and thickness $t$, under a load $F$, is given by standard beam theory (see previous slide) as

$$\delta = F \frac{l^3}{32E_{cell}I},$$

where $E_{cell}$ is the Young’s modulus of the beam material (i.e. the cell wall) and $I$ is the bending moment which is proportional to $t^4$ (recall that $I = wt^3/12$, so for $w=t$, $I = t^4/12$). The force is stress, $\sigma$, multiplied by area, = $F$, i.e.

$$F = \sigma l^2.$$

The strain, $\varepsilon$, is the displacement, $\delta$, divided by the cell length,

$$\varepsilon = \delta / l = F \frac{l^3}{32 E_{cell}I} = F \frac{l^2}{32 E_{cell}I}.$$

Wood: modulus, contd.

- Thus we can obtain Eq. 2 as the ratio of stress to strain.

$$E_{\text{transverse}} = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\left(Fl^3/32E_{\text{cell}}I\right)} = \frac{\sigma}{\left(\sigma l^2 l^3/32 E_{\text{cell}}I\right)} = \frac{32 E_{\text{cell}}}{3} \left(\frac{I}{l^4}\right)$$

for $w=t$, $I = t^4/12$

$$E_{\text{transverse}} = \frac{8}{3} E_{\text{cell}} \left(\frac{t}{l}\right)^4$$
Wood: modulus, contd.

- But we also relate the density to the cell dimensions by writing $\rho \propto (t/l)^2$ and obtain Eq. 2 (where the proportionality constant, $C'' \sim 1$, based on experimental data),

$$E_{\text{transverse}} = C'' E_{\text{cell}} \rho^2.$$

- Note that this derivation is a general one for open-celled foams and happens to be a simple, easy-to-understand approach. Woods have more complex structures than the open cell model which helps to explain the scatter in the data.

- Note that the theory for closed-celled foams (see supplemental slides), which is closer to the actual structure of wood, shows a dependence on $(\rho/\rho_{\text{cell}})^3$, not $(\rho/\rho_{\text{cell}})^2$ as derived here.

Wood: strength

- Here, the story is very similar to that of modulus. The axial modulus is determined by the area fraction of cell wall material, hence the linear dependence on density. The transverse strength, however, is limited by bending and plastic hinge behavior of the cellular structure, hence the quadratic dependence on density. The difference between axial and transverse properties is so great for both modulus and most other mechanical properties that it is always necessary to be aware of the anisotropy of wood, i.e. that the properties vary markedly with direction. More succinctly, wood is much stronger and stiffer along the grain than across the grain. The lower the density, the more obvious the difference.
**Wood: fracture toughness**

\[
\begin{align*}
K_{IC: axial} & \propto K_{IC: cell} \left( \frac{\rho}{\rho_{cell}} \right)^{3/2} \\
K_{IC: transverse} & \propto K_{IC: cell} \left( \frac{\rho}{\rho_{cell}} \right)^{3/2} \\
K_{IC: transverse} & \gg K_{IC: axial}
\end{align*}
\]

- For fracture toughness, the result is given without proof that the cellular structure leads to a 3/2 exponent in the density dependence, regardless of direction. The crucial point is that propagating a crack parallel to the grain is much easier than transverse, by a factor of $\sim 10$! More than one microstructural feature contributes to the high transverse toughness, including fiber pull-out, propagation of secondary cracks perpendicular to the primary crack, and elongation of the polymer chains in the cell walls. Again, there are many different directions and planes for crack propagation in this anisotropic material which further increases the variability of the toughness.

**Wood: moisture content**

- Water is found in wood both in chemically bound form, and stored in vessels (“lumin”).
- The bound form of water strongly affects properties of all kinds.
- The free water has only a minor effect.
- The “fiber saturation point” is the water content that corresponds to saturation of the bound water. The FSP is about 28% of the fully dry wood.
**Bone**

- Similar strong sensitivity of properties to moisture content as observed for wood.
- Dependence of modulus on density is less clear even than for wood.
- Compressive strength varies as the square of the density

Note: bone varies considerably in structure, depending on the local loading that the body puts on it.

**Future Composites**

- Carbon nanotube composites: currently based on polymer-nanotube materials, but combinations of nanotubes with ceramics are being fabricated.
  - (a) Nanotube types
  - (b) TEM micrograph of nanotubes (note fringes in the walls indicating multiple walls); (c) TEM image of multiwalled nanotube (MWNT)-polystyrene thin film composite.

Impact Protection for Space Vehicles

• http://hitf.jsc.nasa.gov/hitfpub/main/index.html
• http://see.msfc.nasa.gov/mod/modtech.htm - shield design.
• http://oea.larc.nasa.gov/PAIS/MISSE.html - materials testing.
• http://www.nasa.gov/lb/missions/science/ spinoff9_nextel_f.html use of Nextel as a shield material.

Summary: Part 2B

• Wood can be understood as a composite material or, more usefully, as a cellular material.
• Wood is a multi-scale composite material.
• The cell walls of wood are themselves composite structures.
• Even the fibers in the cell walls are also composites.
• We can estimate their properties based on the application of beam bending theory to the way in the cell walls deform under load.
• Bone has properties that resemble wood in some respects i.e. a similar dependence of modulus on density.
Part 3

- In this Part, we consider the basic characteristics of fibers for fiber composites.
- We examine how to engineer composite properties by exploiting residual stress.
- We also examine the anisotropy of the properties of composite properties, which builds on what we learned about tensor properties.

Fiber Composites

- An important class of composites is that of fiber composites.
- The materials involved may be metal, ceramic or polymer. Glass-fiber composite is typical in low-cost structures such as boat hulls. Carbon-fiber composites are used in higher performance structures such as airplanes where their higher cost is justified by the requirements. Ceramic composites are used typically for high temperature service, such as heat exchangers.
- The basic idea is to take advantage of high strength and stiffness of the fibers and to obtain damage tolerance (and specific shapes) by embedding them in a suitable matrix. More specifically, the fiber material (e.g. graphite, glass) is a material that would not generally be considered to be a structural material.
- Solid mechanics of fiber composites: the key to understanding the mechanical properties of fiber composites (for fibers whose length is short compared to the size of the component) is load transfer between the matrix and the fibers. This means that the stress on each fiber varies along its length. Also, the composite materials are strongly anisotropic (so tensors are useful again). See discussion in the supplemental slides.
- Modern developments: carbon nanotubes offer exceptional stiffness and strength, not to mention interesting electrical properties in some cases. If we can figure out how to separate out the various different conformations and how to align the nanotubes, there should be a wide range of exciting materials possible.
Fibers for Polymer Matrix Composites

- Many types of fibers are available: carbon, glass, aramid, quartz, polyethylene, boron, silicon carbide, alumina, aluminosilicate.

- The polymer matrix composite business is dominated by volume by carbon, glass and aramid fibers because they offer the best performance:price ratio.

"Mechanics of Fibrous Composites", C.T. Herakovich

Carbon Fibers

- Modulus ranges from 200-750 GPa (compare with steel: 210 GPa)
- Strength ranges from 2-6 GPa
- Breaking strain ranges from 0.2 - 2 %
- Density ranges from 1.76- 2.15
- Highest cost compared to glass or aramid, but greatest range of properties.
- Internal structure consists of radially-aligned graphite platelets, which leads to some anisotropy in properties in the fibers. Both thermal and electrical conductivity are generally good (but then insulation required where metals might be in contact for carbon-fiber composite).
**Glass Fibers**

- Glass fibers produced by spinning liquid glass directly to fine fibers. Just as in the Griffith experiments, the strength is based on small diameter.
- Modulus ranges from 70 - 90 GPa.
- Strength ranges from 1.7-5 GPa
- Breaking strain from 2 to 5%
- Density ~ 2.5 gm/cc.
- “E glass” [electrical, borosilicate glass] is the cheapest and most common. “R glass” and “S glass” is more expensive but more corrosion resistant, for example and higher strength.

**Aramid Fibers**

- Aramide fibers are produced by drawing liquid crystal polymers based on, e.g. polyparabenzamide or polyparaphenylene terephthalamide.
- Polymer chains arranged in radially oriented, kinked sheets. Bonding between the molecules is largely hydrogen bonding so the transverse properties are weak compared to on-axis. Therefore difficult to propagate a crack along a fiber.
- Modulus ranges from 55-120 GPa
- Strength ranges from 3 to 3.6 GPa
- Breaking strain ranges from 2.5 to 4%
- Density ~ 1.45 gm/cc.
- Aramide fibers vulnerable to environmental degradation (sunlight).
Residual Stresses and Composites

- In a stationary body that is free of external loads, the average stress (and moment) must be zero because (Newton’s Laws) there must be no net force on it.
- The stress state inside the body, however, can vary arbitrarily. Such variable internal stresses are often known as *residual stresses* because they are the left-over from previous processing.
- The simplicity of elastic stresses is that they can be superimposed. Therefore one can assume in beam loading that the stresses imposed by external loading can be *added* to the internal variations.
- As with all phenomena, there are engineering applications. Reinforced concrete, for example, is a fiber-reinforced composite with a brittle matrix (concrete) and a ductile fiber reinforcement (steel bars or cable). The steel is typically held in tension during the setting-up of the concrete, resulting in a composite for which the steel is in a state of tension and the concrete is in compression.
- For fiber-reinforced materials, for example, a difference in thermal expansion coefficient can produce a residual stress state in a composite. For example, if the fiber has a smaller CTE and the composite is cooled from a zero stress state at high temperature, then the matrix shrinks more than the reinforcing fibers, putting the matrix in tension and the fibers in compression.
- Safety Glass as commonly used for the windshields of cars rely on residual stress developed through heat treatment. A compressive residual stress near the surface(s) is balanced by a tensile residual stress in the center. Furthermore, the heat treatment is done in such a fashion as to develop a fine pattern so that, if the windshield does break, it shatters into many small but compact pieces that are far less hazardous than the typical shards of window glass.

Reinforced Concrete

- Steps required:
  1. Stretch reinforcing steel cables (i.e. place them in tension)
  2. Pour concrete around the cables; allow concrete to set
  3. Remove tensioning force from steel cables
  4. The steel cables contract elastically but the concrete matrix resists the contraction
  5. Steel remains in tension (did not shrink back to zero strain) whereas the concrete is in compression to balance the tensile stress in the steel cables
- Question: is there an optimum location for the reinforcement within the beam? At the top? Bottom?
- Loading of Reinforced Concrete Beams:
  - As the beam is loaded (e.g. 3-point bending), the concrete underneath the loading point experiences the sum of its residual compressive stress, plus the tensile stress from the bending load. For moderate loads, the stress remains compressive, protecting against brittle failure.
  - The composite is highly anisotropic, of course.
  - Famous example (local to Pittsburgh): the cantilevered terraces of Frank Lloyd Wright’s house, Fallingwater (image above).
  - [http://structsource.com/analysis/types/concrete.htm](http://structsource.com/analysis/types/concrete.htm)
Pre-stressed Reinforced Concrete

Remember: in the absence of external loads (tractions) the net stress in the material must be zero.

\[ 0 = \int \sigma dV \]

\textit{Volume}

Steel rod: large tensile stress from external load

Add concrete, allow to set, no stress in concrete

Remove external load on steel; compressive stress in concrete increases to balance the decreased tensile stress in the steel

Homework Questions

- A worked example is very simple in this case.
- If the fracture toughness, \( K_{IC} \), of concrete is measured to be 2 MPa\(\sqrt{\text{m}} \), and the maximum flaw size is 5 mm (based on the aggregate sizes), what is the maximum tensile stress that it can withstand? Answer: apply the Griffith Eq. with the maximum flaw size as the crack size (since this represents the weak link in the material), which suggests that the breaking stress = \( \sqrt{K_{IC}/\pi c} = \sqrt{2.10^6 / \pi / 5.10^{-3}} \) = 11.28 kPa, which is very small indeed.
- If the volume fraction of reinforcing steel in concrete is limited to 10\%, its yield stress is 1.5 GPa and you can stress the steel to 80\% of its yield (representing the safety factor), what approximate tensile strength can you develop in the concrete via pre-stressing? Answer: assume that you can neglect the inherent tensile strength. Assume that you can apply 1500 * 0.8 MPa tensile stress in the steel, which is balanced by 1500*0.8*0.1/0.9 = 133 MPa compressive stress in the concrete. This residual compressive stress in the concrete represents the maximum tensile stress that you can apply before you expect the concrete to break.
Anisotropy of Cell Wall

6. In lecture 16 we have seen that the cell wall in wood consists of multiple layers, D, S1, S2, and S3, each with fibers oriented in a different way. We would like to mimic nature by fabricating a three-layer planar composite material, consisting of an S2 layer taking up 16% of the thickness, with fibers at 90° to the tensile direction (see sketch below); then an S3 layer taking up 80% of the thickness with fibers oriented at 20° with respect to the tensile axis; the top layer is of the S1 type with fibers oriented at 60°. If an individual layer with fibers parallel to the tensile axis (the x-direction) is described by the following elastic modulus tensor (in engineering notation):

\[
c_{ij} = \begin{pmatrix}
16 & 11 & 11 & 0 & 0 & 0 \\
11 & 16 & 11 & 0 & 0 & 0 \\
11 & 11 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

then what is the Young’s modulus \( E \) of the composite material for a tensile experiment along the \( z \)-direction? (Hint: First, decide whether this case corresponds to the Reuss or Voigt geometry. Then, assuming that the tensile axis lies along the \( z \)-direction, determine what the transformed modulus \( \tilde{c}_{ij} \) looks like for a rotation of \( 90 \) degrees around the \( z \)-axis. You will have to go back to the tensor transformation relation for a fourth rank tensor.)

Cell Wall: Young’s Modulus: Anisotropy

- The first decision is which model to use.
- In this context it means, do we use iso-strain or iso-stress?
- Since we are looking at loading the material in the plane of the layers, then it is appropriate to use the iso-strain model.
- This means that we can use the rule of mixtures for the 3 phases that contribute to the Young’s modulus:
  \[
  \sigma_c = V_1\sigma_1 + V_2\sigma_2 + V_3\sigma_3 = V_1E_1\varepsilon_C + V_2E_2\varepsilon_C + V_3E_3\varepsilon_C.
  \]
- The next step is to compute the moduli.
S in terms of C

In order to compute Young’s modulus, we need to use the reciprocal compliances.

The relationships for s (compliance) in terms of c (stiffness) are symmetrical to those for stiffnesses in terms of compliances (a simple exercise in algebra!).

\[
\begin{align*}
    s_{11} &= \frac{(c_{11}+c_{12})}{(c_{11}-c_{12})(c_{11}+2c_{12})} \\
    &= \frac{(16+11)}{(16-11)(16+22)} \\
    &= 0.1421 \\
    s_{12} &= -\frac{c_{12}}{(c_{11}-c_{12})(c_{11}+2c_{12})} \\
    &= \frac{-11}{(16-11)(16+22)} \\
    &= -0.05789 \\
    s_{44} &= \frac{1}{c_{44}} \\
    &= \frac{1}{1} = 1.
\end{align*}
\]

Rotated compliance (matrix)

- The standard relationship is as follows:

\[
\begin{align*}
    s'_{11} &= s_{11} - \\
    2\left(s_{11} - s_{12} - \frac{1}{2}s_{44}\right)\left\{\alpha_1^2 \alpha_2^2 + \alpha_2^2 \alpha_3^2 + \alpha_3^2 \alpha_1^2\right\}
\end{align*}
\]

- Now we just need to specify the direction cosines, of which only the 1st term, \((\alpha_1\alpha_2)^2\), is non-zero. For the S_3 layer, it is easy because the value is zero, so only s_{11} is used! For S_2 \((\alpha_1\alpha_2)^2 = \cos^2(20) \cos^2(70) = 0.1033\); for S_1 the \((\alpha_1\alpha_2)^2 = \cos^2(60) \cos^2(30) = 0.1875\). The combination of compliances = 2*(0.1421+0.05789-0.5)= -0.3001.
**Compliance values; Young’s Modulus**

- $s_{11}$ for $S_1$: 0.1421
- $s_{11}$ for $S_2$: $0.1421 + 0.1033 \times 0.3001 = 0.1111$
- $s_{11}$ for $S_3$: $0.1421 + 0.1875 \times 0.3001 = 0.08583$
- Make the volume-based average:
  - $1/E_{cell} = 0.1 \times 0.1421 + 0.8 \times 0.1111 + 0.1 \times 0.08585 = 0.111675$
  - $E_{cell} = 1 / 0.111675 = 8.954$

---

**Cell Wall: Young’s Modulus: Anisotropy**

- What if the fibers have, say tetragonal symmetry, as is more likely than cubic? Then the stiffness tensor will take the following form.

$$
c_{ij} = \begin{pmatrix}
16 & 5 & 11 & 0 & 0 & 0 \\
5 & 16 & 11 & 0 & 0 & 0 \\
11 & 11 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}
$$

- Here the challenge is to invert the properties of a tetragonal material so that we ought to use compliances rather than stiffnesses.
**Tetragonal Fibers**

- Let’s further assume that the 4-fold symmetry axis is parallel to the long direction of the fibers.
- Inverting the compliance-stiffness relation, however, is non-trivial for non-cubics. This is found in Nye or Newnham. The relationships are written out for $c$ in terms of $s$, but they are symmetrical so $s$ can be substituted for $c$, and vice versa.

\[
\begin{align*}
\alpha_{11} + \alpha_{12} &= s_{33} / s ; \\
\alpha_{11} - \alpha_{12} &= 1 / (s_{11} - s_{12}) ; \\
\alpha_{13} &= -s_{13} / s_{33} \\
\alpha_{33} &= (s_{11} + s_{12}) / s ; \\
\alpha_{44} &= 1 / s_{44} ; \\
s &= s_{33} (s_{11} + s_{12}) - 2s_{13}^2 .
\end{align*}
\]

- Next we need to find the formulae for the variation in $s_{11}$ with direction.

---

**Tetragonal Fibers, contd.**

- Again, as found in Nye:

\[
s'_{11} = s_{11} \left( \alpha_1^4 + \alpha_2^4 \right) + s_{33} \alpha_3^4 + \left( s_{12} + s_{44} \right) \alpha_1^2 \alpha_2^2 + \alpha_2^2 \left( 1 - \alpha_3^2 \right) \left( s_{13} + s_{44} \right) + \left\{ 2s_{16} \alpha_1 \alpha_2 \left( \alpha_1^2 - \alpha_2^2 \right) \right\}
\]

- The computation is then similar but longer and more detailed.
- What emerges is the conclusion that the cell wall can be stiffer, or more compliant, than is possible by aligning the fibers in only one direction.
**Summary: Part 3**

- In this part, we learned about the properties of *fiber-reinforced composites*.
- We also learned about how important the *anisotropy of composites* often is, and how to represent that anisotropy in terms of tensor properties of materials. Further information on anisotropy of composites can be found in the supplemental slides.

**Part 4**

- In this Part, we consider the basic characteristics of cellular materials.
- We examine the problem of shock absorbing materials as an example of the application of composite properties for foams (cellular materials).
Cellular Materials

- This next section provides some basic information on cellular materials.
- Why study cellular materials? Answer: cellular materials provide a range of properties that are not achievable in bulk materials. Especially when load carrying capacity at very low densities is required, only cellular materials can satisfy the requirements. Shock resistance is also a vital characteristic of cellular materials.
- Cellular structures are feasible (and used for engineering applications) with all materials types. Metal honeycombs are used in transport applications. Ceramic foams are used in insulation. Cellular structures are ubiquitous in biomaterials (wood, bone, shells...).

Honeycombs: properties

- Note the contrast between tension and compression (plateau present), 4.2a vs. 4.2b.
- Even brittle wall materials exhibit progressive failure in compression, 4.2e.
- The stress-strain curves are labeled by their characteristic stages.
- Very important consequences for energy absorbing structures (see later slides)

Fig. 4.2. Compressive and tensile stress-strain curves for honeycombs: (a) and (b) an elastomeric honeycomb; (c) and (d) an elastic-plastic honeycomb; (e) and (f) an elastic–brittle honeycomb.

[Gibson & Ashby: Cellular Materials]
Energy Absorption

- Why are foams useful?! One reason is their capacity to absorb energy.

![Graph](image1)

**Fig. 8.1.** (a) Stress-strain curves for an elastic solid and a foam made from the same solid, showing the energy per unit volume absorbed at a peak stress $\sigma_p$. (b) Energy absorbed per unit volume against peak stress generated by an impact for both foams and the solid from which the foams are made. The foams always absorb more energy than the solid for a given maximum peak stress, $\sigma_p$. Both axes are normalized by the solid modulus, for reasons explained later.

---

**Energy Absorption: 2**

- How do these two graphs connect? Each line on the 2nd graph correspond to a locus of points from the 1st graph, for a particular relative density. Note the turn-over in the curve of energy versus stress: this is the most efficient use of the material.

![Graph](image2)

**Fig. 8.1.** (a) Stress-strain curves for an elastic solid and a foam made from the same solid, showing the energy per unit volume absorbed at a peak stress $\sigma_p$. (b) Energy absorbed per unit volume against peak stress generated by an impact for both foams and the solid from which the foams are made. The foams always absorb more energy than the solid for a given maximum peak stress, $\sigma_p$. Both axes are normalized by the solid modulus, for reasons explained later.
Energy Absorption: 3

During wall buckling, densification proceeds at a approximately constant external stress.

![Diagram showing stress-strain curves for fully densified, wall buckling, and elastic materials.](Gibson)

- Fully Densified
- Wall Buckling
- Elastic

Note that, once the foam starts to densify (steep upturn in the stress-strain curve) then the stress rises with little increase in energy absorbed.

Energy Absorption: 4

- As seen before, the stress-strain (8.4a) can be re-plotted as energy absorbed versus stress (8.4b). Varying the density varies the maximum energy that can be absorbed at the plateau stress.
- We can draw an envelope through the points of maximum energy ÷ plateau stress.
- Variations in other parameters such as strain rate can also be shown on such an energy-stress diagram by plotting only these envelopes.

![Diagram showing energy absorption curves for different densities and strain rates.](Gibson)
### Shock Cushions

- Once one knows the energy-stress characteristic of a material, it is possible to calculate the **optimum thickness**.
- Given the kinetic energy to be absorbed, \(U\), and the area of contact between object and foam, \(A\), the thickness, \(t\), is given by

\[
t = \frac{U}{W A} \quad (Eq. 1)
\]

where \(W\) is the energy absorbed per unit volume in the foam.
- Typically, the mass of the object, \(m\), and the peak deceleration, \(a\), is also specified (as a multiple of gravitational acceleration, \(g\)) which determines the maximum stress, \(\sigma\),

\[
\sigma = \frac{ma}{A} \quad (Eq. 2)
\]

### Shock Cushion: 2

- In addition, a drop height is specified which in turn sets the velocity, \(v\), and the energy, \(U\), that must be absorbed; \(U = m v^2 / 2\). Thus the thickness, \(t\), is given by

\[
t = \frac{mv^2}{2WA} \quad (Eq. 3)
\]

- This in turn specifies the strain rate, \(d\varepsilon/dt\), in the foam which affects the energy-stress relationship (see Fig. 8.4c):

\[
d\varepsilon/dt = \frac{v}{t} \quad (Eq. 4)
\]

- A good place to start is to identify the maximum allowable stress and read off the associated energy at a high strain rate. The energy is, however, a function of both stress and strain rate, so some iteration is required to identify a suitable thickness.
**Shock Cushion: 3**

**Worked Example**

**Problem specification**

- Mass of packaged object: 500 gms.
- Area of contact between object and foam: \( A = 0.01 \text{ m}^2 \)
- Velocity of package on impact, \( v = 4.5 \text{ m/s} \) (drop height, \( h = 1 \text{ m} \))
- Energy to be absorbed, \( U = \frac{mv^2}{2} = 5 \text{ J} \)
- Max. allowable force on package (10g deceleration), \( F = ma = 50 \text{ N} \)
- Max. allowable peak stress (Eq. 2), \( \sigma_p = \frac{F}{A} = 5 \text{ kPa} \)
- Solid modulus of polyurethane foam, \( E_s = 50 \text{ MPa} \)
- Max. allowable peak stress, normalized = \( \frac{\sigma_p}{E_s} = 0.0001 \)

We use Gibson-Ashby, fig. 8.8 (next slide).

Gibson & Ashby: Table 8.2, p. 231

---

**Shock Cushion: 4**

To start working on the problem, we have to make some rather arbitrary choices of thickness that bracket the likely result.

- Choice of thickness, \( t \): 1 m, 0.001 m
- Strain rate, \( \frac{d\varepsilon}{dt} = \frac{v}{t} \) (Eq 4): 4.5 s\(^{-1}\), 4500 s\(^{-1}\)
- Energy/modulus \((W/E_s)\) at \( \frac{\sigma_p}{E_s} = 0.0001 \):
  - (Fig. 8.8) \( 5.25 \times 10^{-5} \), \( 7.4 \times 10^{-5} \)
- Energy absorbed/unit volume: 2.62 kJ/m\(^3\), 3.70 kJ/m\(^3\)

To complete the problem, we have to iterate on the thickness until we converge on a self-consistent result.
To continue with the problem, we re-calculate the thicknesses from Eq. 1.

- **Thickness,** \( t = \frac{U}{WA} \):
  - 0.19 m
  - 0.14 m

- **Strain rate,** \( \frac{de}{dt} = \frac{v}{t} \) (Eq 4):
  - 24 s\(^{-1}\)  
  - 32 s\(^{-1}\)

- **Energy/modulus** \( \frac{W}{E_s} \) at \( \frac{\sigma_p}{E_s} = 0.0001 \):
  - (Fig. 8.8)
    - 6.6 \(10^{-5}\)  
    - 6.7 \(10^{-5}\)

- **Energy absorbed/unit volume:** 3.30 kJ/m\(^3\)  
  - 3.35 kJ/m\(^3\)

Clearly we have nearly converged, so we have to iterate on the thickness one more time, using \( t = \frac{U}{WA} \), which gives \( t = 150\) mm and an optimum relative density = 0.01.

---

**Summary: Part 4**

- Foams or cellular materials are an example of composite materials.
- We developed an example of how cellular materials are useful as shock cushions.
- This lead to worked example of how calculate the optimum thickness of such as shock cushion.
Summary: Overall

- Composite materials have been described with respect to their microstructure-property relationships.
- Use of the composite approach enables much larger variations in properties to be achieved within a given material type.
- Careful optimization of the material with respect to all the property requirements [for a given application] is essential.
- CTE of a composite can be estimated (supplementary slides) from the CTEs of the constituent phases.

References

- Structural Materials, Butterworth Heinemann, edited by G. Weidmann, P. Lewis and N. Reid.
Supplemental Slides

- The following slides contain supplemental material that will be of interest to those who are curious to obtain more detail.

Improved bounds

- Upper and lower bounds for modulus have been developed by Hashin & Shtrikman that narrow the range between the two bounds.
- Different formulae established for bulk, $K$, and shear moduli, $G$.
- Notation: bulk moduli $K_A$ and $K_B$; shear moduli $G_A$ and $G_B$.

$$K_{lower} = K_A + \frac{V_B}{1 + \frac{3(1 - V_B)}{K_B - K_A} + \frac{3K_A + 4G_A}{3K_A + 4G_A}}$$
Hashin-Shtrikman

\[
K_{\text{upper}} = K_B + \frac{1 - V_B}{1 - \frac{K_A - K_B}{K_B}} + \frac{3V_B}{3K_B + 4G_B}
\]

\[
G_{\text{lower}} = G_A + \frac{V_B}{G_B - G_A} + \frac{6(K_A + 2G_A)(1 - V_B)}{5G_A(3K_A + 4G_A)}
\]

\[
G_{\text{upper}} = G_B + \frac{1 - V_B}{G_A - G_B} + \frac{6(K_B + 2G_B)V_B}{5G_A(3K_B + 4G_B)}
\]

Examples

- This example from Green’s text shows how the bulk and shear moduli vary with volume fraction for two phases whose moduli differ by a factor of 10.
- The result shows that the H-S bounds are generally more useful.
Anisotropy in Composites

- The same methods developed in lecture 4 for describing the anisotropy of single crystals can be applied to composites.
- Anisotropy is important in composites, not because of the intrinsic properties of the components but because of the arrangement of the components.
- As an example, consider (a) a uniaxial composite (e.g. tennis racket handle) and (b) a flat panel cross-ply composite (e.g. wing surface).

Fiber Symmetry
Fiber Symmetry

- We will use the same *matrix notation* for stress, strain, stiffness and compliance as for single crystals.
- The compliance matrix, \( \mathbf{s} \), has 5 independent coefficients.

\[
\mathbf{s} = \begin{bmatrix}
    s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
    s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\
    s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\
    0 & 0 & 0 & s_{44} & 0 & 0 \\
    0 & 0 & 0 & 0 & s_{44} & 0 \\
    0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \\
\end{bmatrix}
\]

Relationships

- For a uniaxial stress along the z (3) direction,

\[
E_3 = \frac{\sigma_3}{\varepsilon_3} = \frac{1}{s_{33}} \left( \frac{\sigma_{zz}}{\varepsilon_{zz}} \right)
\]

- This stress causes strain in the transverse plane:

\( e_{11} = e_{22} = s_{12} \sigma_{33} \). Therefore we can calculate Poisson’s ratio as:

\[
\nu_{13} = \frac{e_1}{\varepsilon_3} = \frac{s_{13}}{s_{33}} \left( \frac{e_{xx}}{\varepsilon_{zz}} \right)
\]

- Similarly, stresses applied perpendicular to z give rise to different moduli and Poisson’s ratios.

\[
E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{1}{s_{11}}, \quad \nu_{21} = -\frac{s_{12}}{s_{11}}, \quad \nu_{31} = -\frac{s_{13}}{s_{11}}
\]
**Relationships, contd.**

- Similarly the torsional modulus is related to shears involving the z axis, i.e. $yz$ or $xz$ shears:
  \[ s_{44} = s_{55} = 1/G \]

- Shear in the x-y plane is related to the other compliance coefficients:
  \[ s_{66} = 2(s_{11} - s_{12}) = 1/G_{xy} \]

---

**Plates: Orthotropic Symmetry**

- Again, we use the same *matrix notation* for stress, strain, stiffness and compliance as for single crystals.

- The compliance matrix, $s$, has 9 *independent coefficients*.

\[
\begin{bmatrix}
  s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
  s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\
  s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & s_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & s_{55} & 0 \\
  0 & 0 & 0 & 0 & 0 & s_{66}
\end{bmatrix}
\]
**Plates: 0° and 90° plies**

- If the composite is a laminate composite with fibers laid in at 0° and 90° in equal thicknesses then the symmetry is higher because the x and y directions are equivalent.
- The compliance matrix, \( s \), has 6 independent coefficients.

\[
\begin{bmatrix}
  s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
  s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\
  s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & s_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & s_{44} & 0 \\
  0 & 0 & 0 & 0 & 0 & s_{66}
\end{bmatrix}
\]

**Anisotropy: Practical Applications**

- The practical applications of anisotropy of composites, especially fiber-reinforced composites are numerous.
- The stiffness of fiber composites varies tremendously with direction. Torsional rigidity is very important in car bodies, boats, aeroplanes etc.
- Even in monolithic polymers (e.g. drawn polyethylene) there exists large anisotropy because of the alignment of the long-chain molecules.
Closed Cell Wall Bending

- LHS: response to compressive loading in the x direction;
- RHS: response to compressive loading in the y direction.
- Consider loading in the x direction: each oblique segment experiences bending at each end. The load, $P$, is
  $$P = \sigma_1(h + l \sin \theta)b$$
  - see fig. 4.8b

![Fig. 4.8. Cell deformation by cell wall bending, giving linear-elastic extension or compression of the honeycomb: (a) the undeformed honeycomb; (b) and (c) the bending caused by loads in the $x_1$ and $x_2$ directions.

[Gibson: Cellular Materials]

Modulus(relative density)

- Treat each segment as a beam of length $l$, thickness $t$, depth $b$, and Young’s Modulus $E_s$.
- The force, $C$, resolved on the $y$ (vertical) direction must be zero in order to satisfy equilibrium.
- The moment, $M$, on the segment:
  $$M = Pl \sin \theta / 2$$
- The deflection, $\delta$, of the segment:
  $$\delta = Pl^3 \sin \theta / 12E_{cell}I$$
  where $I$ is the second moment of inertia:
  $$I = bt^3 / 12$$
**Cell Geometry (general hexagonal)**

\[ \text{relative density} = \frac{\rho^*}{\rho_s} = \frac{(t/l)(h/l + 2)}{2 \cos \theta (h/l + \sin \theta)} \]

\[ \theta \]

- **Modulus (relative density):** \( E_1 \)

- We need the component of the deflection that is parallel to the X axis, \( \delta \sin \theta \). Thus the strain is:

\[ \varepsilon_1 = \frac{\delta \sin \theta}{l \cos \theta} = \frac{\sigma_1 (h + l \sin \theta) b l^2 \sin^2 \theta}{12 E_s I \cos \theta} \]

\[ E_1 = \frac{\sigma_1}{\varepsilon_1} \quad \therefore \quad \frac{E_1}{E_s} = \left( \frac{t}{l} \right)^3 \frac{\cos \theta}{(h/l + \sin \theta) \sin^2 \theta} \]
Modulus (relative density): $E_2$

- The modulus in the perpendicular direction is similar.

$$\varepsilon_2 = \frac{\delta \cos \theta}{h + l \sin \theta} = \frac{\sigma_2 b l^4 \cos^4 \theta}{12 E_s I (h + l \sin \theta)}$$

$$E_2 = \frac{\sigma_2}{\varepsilon_2} \quad \therefore \frac{E_2}{E_s} = \left(\frac{t}{l}\right)^3 \left(\frac{h/l + \sin \theta}{\cos^3 \theta}\right)$$

Modulus (relative density): regular hex

For regular hexagons, the reduced moduli in the two directions are the same:

$$E_1 / E_{cell} = E_2 / E_{cell} = 2.3 (t/l)^3$$

We already established that the relative density for a regular hexagon is $2/\sqrt{3} (t/l) \sim 2.3 (t/l)$, so we can write:

$$E_1 / E_{cell} = E_2 / E_{cell} = 2.3 \left(\frac{\rho}{\rho_{cell}}\right)^3$$
Wood Deformation

Moisture, CTE

Table 2.4 Coefficient of Linear Thermal Expansion of Various Woods and Other Materials Per Degree Centigrade

<table>
<thead>
<tr>
<th>Material</th>
<th>Longitudinal $10^{-5}$</th>
<th>Transverse $10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Norried</em> <em>abies</em></td>
<td>5.41</td>
<td>34.1</td>
</tr>
<tr>
<td><em>Phoenix sphenoba</em></td>
<td>4.00</td>
<td>72.7</td>
</tr>
<tr>
<td><em>Quercus robur</em></td>
<td>4.92</td>
<td>54.4</td>
</tr>
<tr>
<td>GRP, 60/40, unidirectional</td>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>CFRP, 60/40, unidirectional</td>
<td>10.0</td>
<td>1.00</td>
</tr>
<tr>
<td>Mild steel</td>
<td>12.6</td>
<td></td>
</tr>
<tr>
<td>Duralumin</td>
<td>22.5</td>
<td></td>
</tr>
<tr>
<td>Nylon 6/6</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>Polypropylene</td>
<td>110</td>
<td></td>
</tr>
</tbody>
</table>
Strength of Fiber Composites

• Just as for modulus, the simplest model for composite strength is the Rule of Mixtures, where $\sigma_m$ is the tensile strength of the matrix.

$$\sigma_c = \sigma_m V_m + \sigma_f V_f$$

• A better model takes account of the actual stress-strain characteristics of the component phases.

• In MMCs, for example, the fiber reinforcement is often quite brittle compared to the matrix (e.g. graphite fibers in Mg, SiC fibers in Ti).

• The brittleness of the fibers limits the strain that can be applied to a composite.
Ductile matrix + brittle fibers

- If the composite is deformed beyond the breaking strain of the fibers, then the broken fibers no longer support load and their strengthening contribution is lost. In this case, the strength is just this:

\[ \sigma_c = \sigma_m V_m \]

Ductile matrix + brittle fibers, contd.

- At high enough volume fractions, however, the hardening in the matrix is exhausted before the failure strength of the fibers is reached. The matrix then fails at a (constant) stress,

\[ \sigma^*_m = E_m \varepsilon_f^* \]

which corresponds to the failure strain, \( \varepsilon_f^* \), of the fibers. Under these conditions, the strength of the composite is an average of the strength of the fibers and the strength of the matrix at the failure strain of the fibers. The strength of the composite then increases with volume fraction of reinforcing fibers and is given by:

\[ \sigma_c = \sigma^*_m V_m + \sigma_f V_f \]


**Ductile matrix + brittle fibers, contd.**

- Thus there is a cross-over between the two types of behavior.
- A minimum volume fraction of fibers is required in order for the strength of the fiber composite to exceed that of the matrix.

\[ \sigma_c = \sigma^*_m V_m + \sigma_f V_f \]

\[ \sigma_c = \sigma_m V_m \]

---

**Coefficient of Thermal Expansion**

- The next section relates the coefficient of thermal expansion (CTE) to the microstructure of composites, using glass-ceramics as an example.
CTE versus modulus

- The thermal expansion coefficient of a composite, $\alpha_{comp}$, can be related to the expansion coefficients and bulk moduli of the constituent phases by the following. Obviously, the composite bulk modulus must be determined by other means.

$$\alpha_{composite} = \alpha_A + \frac{K_B (\alpha_B - \alpha_A) (K_A - K_{composite})}{K_{composite} (K_A - K_B)}$$

Quartz

- The compressibility for cristobalite is given as $100.10^{-6}$ K$^{-1}$ (alpha-cristobalite) and $4.8.10^{-6}$ K$^{-1}$ (beta-cristobalite).
- The CTE is given as $25.2.10^{-6}$ for alpha-cristobalite and $11.2.10^{-6}$ for beta-cristobalite.
- Compare to the range of $12-20.10^{-6}$ K$^{-1}$ claimed for the glass-ceramic.

Cristobalite structure: [Chiang et al.]
Li-Zn glass ceramics

- Note the variation in expansion at the alpha-beta transition (displacive) in cristobalite.