numbers appear if ι is fixed. For example, the analogue of the de Moivre formula is true: $(\cos(\theta) + \iota \sin(\theta))^k = \cos(k\theta) + \iota \sin(k\theta)$. ☒

$\boxed{\boxdot}$ *Quaternion components ↔ Euler angles*
Using (2.33), the quaternion components can be expressed through Euler angles

$$\begin{aligned} q^0 &= \cos(\phi/2)\cos((\varphi_1+\varphi_2)/2) \ , \quad & q^1 &= -\sin(\phi/2)\cos((\varphi_1-\varphi_2)/2) \ , \\ q^2 &= -\sin(\phi/2)\sin((\varphi_1-\varphi_2)/2) \ , \quad & q^3 &= -\cos(\phi/2)\sin((\varphi_1+\varphi_2)/2) \ . \end{aligned}$$

Again, the quaternion with the opposite sign represents the same rotation. From these relations we obtain the inverse formulas for the calculation of the Euler angles from the quaternion. Let $\chi := [((q^0)^2+(q^3)^2)((q^1)^2+(q^2)^2)]^{1/2} = (\sin\phi)/2$. If $\chi \neq 0$,

$$\begin{aligned} &\cos\phi = \left((q^0)^2+(q^3)^2\right) - \left((q^1)^2+(q^2)^2\right) \ , \\ &\cos\varphi_1 = (-q^0q^1 - q^2q^3)/\chi \ , & &\sin\varphi_1 = (-q^0q^2 + q^1q^3)/\chi \ , \\ &\cos\varphi_2 = (-q^0q^1 + q^2q^3)/\chi \ , & &\sin\varphi_2 = (+q^0q^2 + q^1q^3)/\chi \ . \end{aligned}$$

If $\chi = 0$, there occurs $q^1 = 0 = q^2$ or $q^0 = 0 = q^3$. In the first case, we have

$$\phi = 0 \ , \ \cos(\varphi_1+\varphi_2) = (q^0)^2 - (q^3)^2 \ , \ \sin(\varphi_1+\varphi_2) = -2q^0q^3 \ .$$

In the second one, the angles are given by

$$\phi = \pi \ , \ \cos(\varphi_1-\varphi_2) = (q^1)^2 - (q^2)^2 \ , \ \sin(\varphi_1-\varphi_2) = 2q^1q^2 \ .$$

Both instances correspond to singularities of Euler angles and only combinations of φ_1 and φ_2 are determinable. ☒

$\boxed{\boxdot}$ *Quaternionic form of transformation of a vector*
Based on (2.41), the transformation of Cartesian coordinates $x'_i = O_{ij}x_j$ can be expressed as

$$x'_i = \left((q^0)^2 - q^kq^k\right)x_i + 2q^iq^jx_j - 2\varepsilon_{ijk}q^0x_jq^k \ .$$

The same form has the i-th quaternion component of the product qxq^* where x is the "vector" quaternion $x = 0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. Thus, with $x' = x'_ie_i$ $(i = 1, 2, 3)$, the transformation can be written as

$$x' = q \diamond x \diamond q^{-1} = q \diamond x \diamond q^* \ . \tag{2.49}$$

This rule follows also from the interpretation of quaternions as elements of the Clifford algebra $\mathcal{C}_3^+$ and (1.20).

The isomorphism between the group of unit quaternions and $SU(2)$ and the fact that the quaternions e_μ correspond to ς_μ matrices allows us to write

$$X' = UXU^\dagger \ ,$$

where the unitary matrix U is related to q by (2.36), $X = x_i\varsigma_i$ and $X' = x'_i\varsigma_i$. These vector transformation rules are similar to Eq.(2.16). ☒

$\boxed{\boxdot}$ *Improper rotations and quaternions*
One may ask about the possibility to represent improper rotations by quaternions. With quaternion algebra seen as a standing alone construct, it is natural to