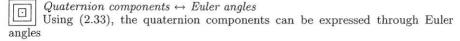
numbers appear if  $\iota$  is fixed. For example, the analogue of the de Moivre formula is true:  $(\cos(\theta) + \iota \sin(\theta))^k = \cos(k\theta) + \iota \sin(k\theta)$ .  $\boxtimes$ 



$$\begin{array}{ll} q^0 = \cos(\phi/2)\cos((\varphi_1+\varphi_2)/2) \;, & q^1 = -\sin(\phi/2)\cos((\varphi_1-\varphi_2)/2) \;, \\ q^2 = -\sin(\phi/2)\sin((\varphi_1-\varphi_2)/2) \;, & q^3 = \cos(\phi/2)\sin((\varphi_1+\varphi_2)/2) \;. \end{array}$$

Again, the quaternion with the opposite sign represents the same rotation. From these relations we obtain the inverse formulas for the calculation of the Euler angles from the quaternion. Let  $\chi := [((q^0)^2 + (q^3)^2)((q^1)^2 + (q^2)^2)]^{1/2} = (\sin \phi)/2$ . If  $\chi \neq 0$ ,

$$\begin{split} \cos\phi &= \left( (q^0)^2 + (q^3)^2 \right) - \left( (q^1)^2 + (q^2)^2 \right) \;\;, \\ \cos\varphi_1 &= \left( -q^0q^1 + q^2q^3 \right) / \chi \;\;, & \sin\varphi_1 &= \left( -q^0q^2 + q^1q^3 \right) / \chi \;\;, \\ \cos\varphi_2 &= \left( -q^0q^1 + q^2q^3 \right) / \chi \;\;, & \sin\varphi_2 &= \left( +q^0q^2 + q^1q^3 \right) / \chi \;\;. \end{split}$$

If  $\chi = 0$ , there occurs  $q^1 = 0 = q^2$  or  $q^0 = 0 = q^3$ . In the first case, we have

$$\phi = 0$$
,  $\cos(\varphi_1 + \varphi_2) = (q^0)^2 - (q^3)^2$ ,  $\sin(\varphi_1 + \varphi_2) = 2q^0q^3$ .

In the second one, the angles are given by

$$\phi = \pi$$
,  $\cos(\varphi_1 - \varphi_2) = (q^1)^2 - (q^2)^2$ ,  $\sin(\varphi_1 - \varphi_2) = 2q^1q^2$ .

Both instances correspond to singularities of Euler angles and only combinations of  $\varphi_1$  and  $\varphi_2$  are determinable.  $\boxtimes$ 

Quaternionic form of transformation of a vector Based on (2.41), the transformation of Cartesian coordinates  $x'_i = O_{ij}x_j$  can be expressed as

$$x'_{i} = \left( (q^{0})^{2} - q^{k} q^{k} \right) x_{i} + 2q^{i} q^{j} x_{j} - 2\varepsilon_{ijk} q^{0} x_{j} q^{k}$$
.

The same form has the *i*-th quaternion component of the product  $qxq^*$  where x is the "vector" quaternion  $x = 0e_0 + x_1e_1 + x_2e_2 + x_3e_3$ . Thus, with  $x' = x_i'e_i$  (i = 1, 2, 3), the transformation can be written as

$$x' = q \diamond x \diamond q^{-1} = q \diamond x \diamond q^* . \tag{2.49}$$

This rule follows also from the interpretation of quaternions as elements of the Clifford algebra  $C_3^+$  and (1.20).

The isomorphism between the group of unit quaternions and SU(2) and the fact that the quaternions  $e_{\mu}$  correspond to  $\varsigma_{\mu}$  matrices allows us to write

$$X' = UXU^{\dagger} \ ,$$

where the unitary matrix U is related to q by (2.36),  $X = x_i \varsigma_i$  and  $X' = x'_i \varsigma_i$ . These vector transformation rules are similar to Eq.(2.16).  $\boxtimes$ 

Improper rotations and quaternions
One may ask about the possibility to represent improper rotations by quaternions. With quaternion algebra seen as a standing alone construct, it is natural to