

Objective

Properties

Vectorz.

Taylor-  
factor

Sngl.-slip

RC model

# *Plastic Anisotropy: Relaxed Constraints, Theoretical Textures*

Texture, Microstructure & Anisotropy

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# *Objective*

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- The objective of this lecture is to complete the description of plastic anisotropy.
- Vectorization of stress and strain
- Definition of Taylor factor
- Comparison of single with multiple slip
- Description of Relaxed Constraints

# References

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RC model

- Kocks, Tomé & Wenk: *Texture & Anisotropy* (Cambridge); chapter 8, 1996. Detailed analysis of plastic deformation and texture development.
- Reid: *Deformation Geometry for Materials Scientists*, 1973. Older text with many nice worked examples. Be careful of his examples of calculation of Taylor factor because, like Bunge & others, he does not use von Mises equivalent stress/strain to obtain a scalar value from a multiaxial stress/strain state.
- Hosford: *The Mechanics of Crystals and Textured Polycrystals*, 1993 (Oxford). Written from the perspective of a mechanical metallurgist with decades of experimental and analytical experience in the area.
- Khan & Huang: *Continuum Theory of Plasticity*. Written from the perspective of continuum mechanics.
- De Souza Neto, Peric & Owen: *Computational Methods for Plasticity*, 2008 (Wiley). Written from the perspective of continuum mechanics.

## *Vectorization of stress, strain*

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- The lack of dependence on hydrostatic stress and volume constancy permits a different vectorization to be used (as compared with Voigt/matrix notation for anisotropic elasticity). The following set of basis tensors provides a systematic approach. The first tensor,  $\mathbf{b}^{(1)}$  represents deviatoric tension on Z, second plane strain compression in the Z plane, and the second row are the three simple shears in the  $\{XYZ\}$  system. The first two tensors provide a basis for the  $\pi$ -plane.

## Vectorization: 2

Objective	$\mathbf{b}^{(1)} \equiv \frac{1}{\sqrt{6}} \begin{pmatrix} \bar{1} & 0 & 0 \\ 0 & \bar{1} & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\mathbf{b}^{(2)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
Properties			
Vectorz.			
Taylor-factor	$\mathbf{b}^{(3)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbf{b}^{(4)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\mathbf{b}^{(5)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Sngl.-slip			
RC model			

The components of the vectorized stresses and strains are then contractions (projections) with the basis tensors:

$$\sigma_\lambda = \boldsymbol{\sigma} : \mathbf{b}^{(\lambda)}, \quad \varepsilon_\lambda = \boldsymbol{\varepsilon} : \mathbf{b}^{(\lambda)}$$

There is, in fact, a 6<sup>th</sup> eigentensor that separates out the hydrostatic components of stress and strain

## Vectorization: 3

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- This preserves *work conjugacy*, i.e.  $\sigma:\varepsilon = \sum_{\lambda} \sigma_{\lambda} \varepsilon_{\lambda}$ . The Lequeu vectorization scheme was almost the same as this but with the first two components interchanged. It is also useful to see the vector forms in terms of the regular tensor components.

$$S_{\lambda} = \left( \frac{2\sigma_{33} - \sigma_{11} - \sigma_{22}}{\sqrt{6}}, \frac{\sigma_{22} - \sigma_{11}}{\sqrt{2}}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{31}, \sqrt{2}\sigma_{12} \right)$$

$$D_{\lambda} = \left( \frac{2D_{33} - D_{11} - D_{22}}{\sqrt{6}}, \frac{D_{22} - D_{11}}{\sqrt{2}}, \sqrt{2}D_{23}, \sqrt{2}D_{31}, \sqrt{2}D_{12} \right)$$

## *Maximum work, summary*

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- The sum of the shears for the actual set of active systems is less than any hypothetical set (Taylor's hypothesis). This also shows that we have obtained an *upper* bound on the stress required to deform the crystal because we have approached the solution for the work rate from above: any hypothetical solution results in a *larger* work rate than the actual solution.

# *Taylor factor, $M$*

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RC model

- We can take ratios of stresses, or strains to define the Taylor factor,  $M$ ; if a simple choice of deformation axes is made, such as uniaxial tension, then the indices can be dropped to obtain the typical form of the equation,  $\sigma = \langle M \rangle \tau_{crss}$ . For the polycrystal, the arithmetic mean of the Taylor factors is typically used to represent the ratio between the macroscopic flow stress and the critical resolved shear stress. This relies on the assumption that weak (low Taylor factor) grains cannot deform much until the harder ones are also deforming plastically, and that the harder (high Taylor factor) grains are made to deform by a combination of stress concentration and work hardening around them.



# Taylor factor, multiaxial stress

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- For multiaxial stress states, one may use the effective stress, e.g. the von Mises stress (defined in terms of the stress deviator tensor,  $\mathbf{S} = \boldsymbol{\sigma} - (\sigma_{ii}/3)\mathbf{I}$ , and also known as *effective stress*). Note that the equation below provides the most self-consistent approach for calculating the Taylor factor for multi-axial deformation because it ties everything back to the yield stress in uniaxial tension.

$$\sigma_{vonMises} \equiv \sigma_{vM} = \sqrt{\frac{3}{2} \mathbf{S} : \mathbf{S}}$$

$$M = \frac{\sigma_{vM}}{\tau} = \frac{\sum_s \Delta \gamma^{(s)}}{d\varepsilon_{vM}} = \frac{dW}{\tau_c d\varepsilon_{vM}} = \frac{\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}}{\tau_c d\varepsilon_{vM}}$$

# Taylor factor, multiaxial strain

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- Similarly for the von Mises equivalent strain increment (where  $d\varepsilon_p$  is the plastic strain increment which has zero trace, i.e.  $d\varepsilon_{ii}=0$ ).

$$d\varepsilon_{vonMises} \equiv d\varepsilon_{vM} = \sqrt{\frac{2}{3} d\varepsilon_p : d\varepsilon_p} = \frac{2}{\sqrt{3}} \sqrt{\frac{1}{2} d\varepsilon_{ij} : d\varepsilon_{ij}} =$$

$$\sqrt{\left(\frac{2}{9}\right) \left\{ (d\varepsilon_{11} - d\varepsilon_{22})^2 + (d\varepsilon_{22} - d\varepsilon_{33})^2 + (d\varepsilon_{33} - d\varepsilon_{11})^2 \right\} + \frac{4}{3} \left\{ d\varepsilon_{23}^2 + d\varepsilon_{31}^2 + d\varepsilon_{12}^2 \right\}}$$

$$M = \frac{\sigma_{vM}}{\tau} = \frac{\sum_s \Delta\gamma^{(s)}}{d\varepsilon_{vM}} = \frac{dW}{\tau_c d\varepsilon_{vM}} = \frac{\sigma : d\varepsilon}{\tau_c d\varepsilon_{vM}}$$

Compare with single slip: Schmid factor =  $\cos\phi\cos\lambda = \tau/\sigma$

# *Uniaxial compression/tension*

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RC model

- So, for axisymmetric straining paths, the orientation dependence within a Standard Stereographic Triangle (SST) is as follows. The velocity gradient has the form (upper +/- means compression, lower +/- means tension):

$$L = \begin{pmatrix} \pm\delta & 0 & 0 \\ 0 & \pm\delta & 0 \\ 0 & 0 & \mp 2\delta \end{pmatrix}$$

The von Mises equivalent strain for such a tensile strain is always  $2\delta$ .

# Taylor factor(orientation)

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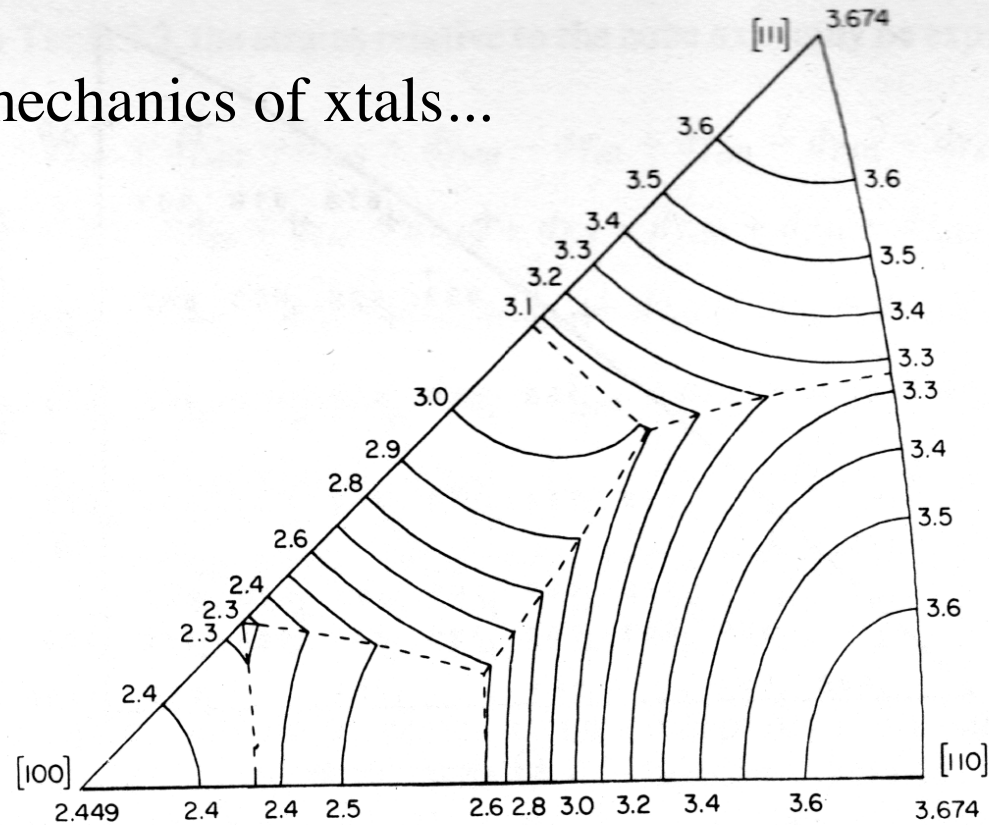
Vectorz.

Taylor-  
factor

Sngl.-slip

RC model

Hosford: mechanics of xtals...



**Fig. 3.4.** Orientation dependence of  $M$  for axisymmetric flow. (From Chin and Mammel [7].)

## *Texture hardening*

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RC model

- Note that the Taylor factor is largest at both the 111 and 110 positions and a minimum at the 100 position. Thus a cubic material with a perfect  $\langle 111 \rangle$  or  $\langle 110 \rangle$  fiber will be 1.5 times as strong in tension or compression as the same material with a  $\langle 100 \rangle$  fiber texture. This is not as dramatic a strengthening as can be achieved by other means, e.g. precipitation hardening, but it is significant. Also, it can be achieved without sacrificing other properties.

# *Uniaxial deformation: single slip*

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Vectorz.

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RC model

- Recall the standard picture of the single crystal under tensile load. In this case, we can define angles between the tensile direction and the slip plane normal,  $\phi$ , and also between the tensile direction and the slip direction,  $\lambda$ . Given an applied tensile stress (force over area) on the crystal, we can calculate the shear stress resolved onto the particular slip system as  $\tau = \sigma \cos\phi \cos\lambda$ . This simple formula (think of using only the direction cosine for the slip plane and direction that corresponds to the tensile axis) then allows us to rationalize the variation of stress with testing angle (crystal orientation) with Schmid's Law concerning the existence of a critical resolved shear stress (CRSS).

# Single slip: Schmid factors

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RC model

- Re-write the relationship in terms of (unit) vectors that describe the slip plane,  $\mathbf{n}$ , and slip direction,  $\mathbf{b}$ : index notation and tensor notation are used interchangeably.

$$\tau = \hat{b} \sigma \hat{n}$$

$$\tau = \sigma_{ij} b_i n_j$$

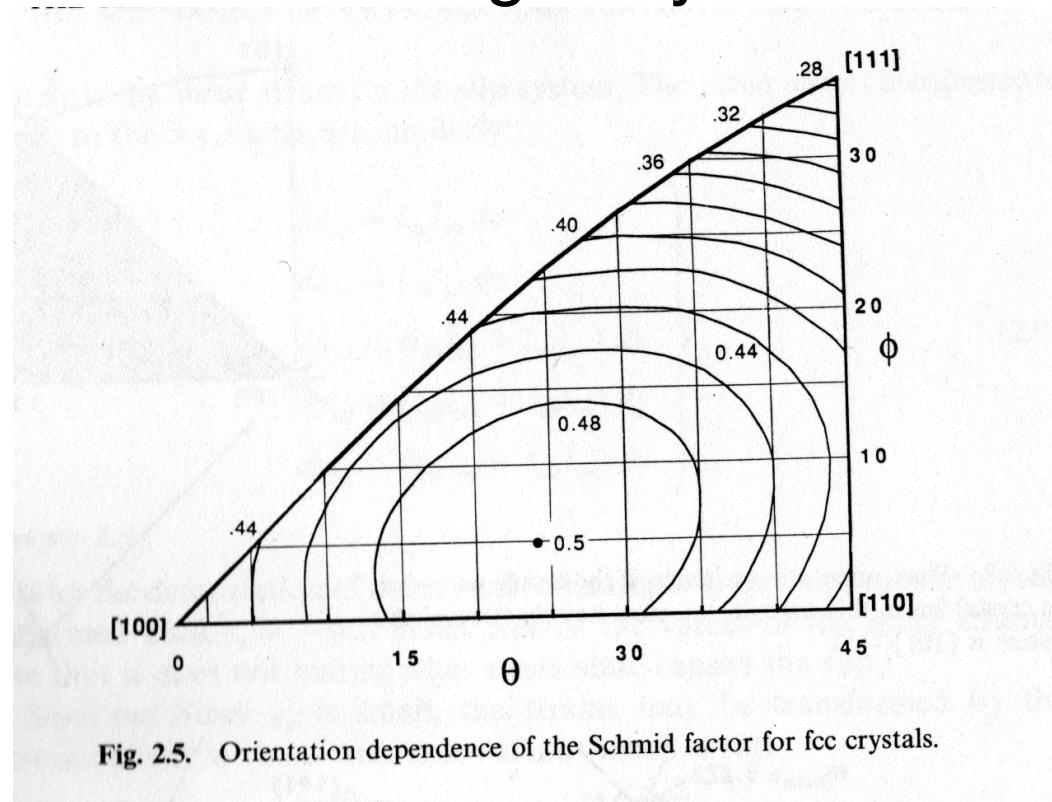


Fig. 2.5. Orientation dependence of the Schmid factor for fcc crystals.



## *Single: multiple comparison*

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RC model

- It is interesting to consider the difference between multiple slip and single slip stress levels because of its relevance to deviations from the Taylor model. Hosford presents an analysis of the *ratio* between the stress required for multiple slip and the stress for single slip under axisymmetric deformation conditions (in  $\{111\}\langle 110\rangle$  slip).



# Single: Multiple comparison

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Max. difference (=1.65)  
between single &  
multiple slip

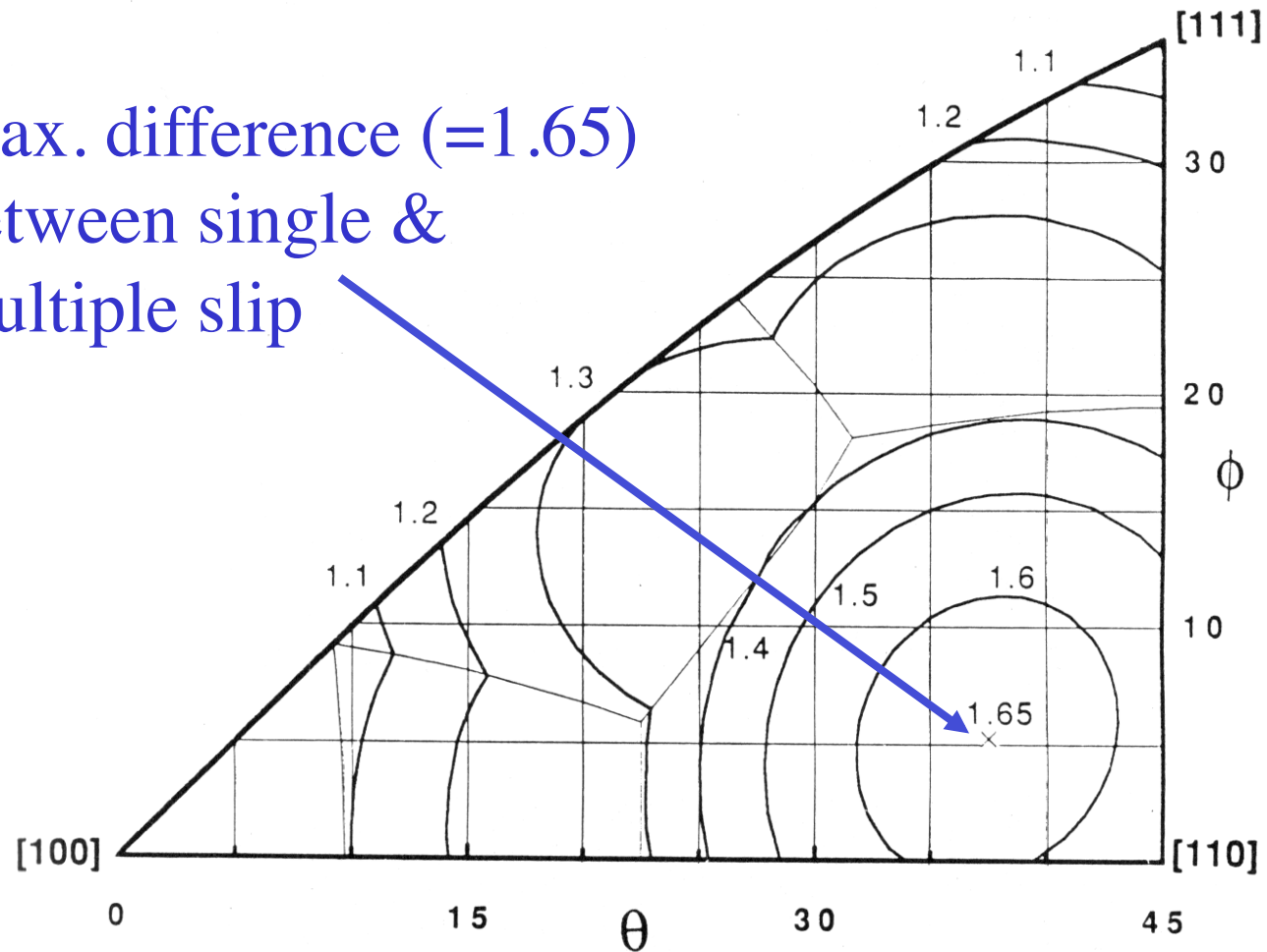


Figure 6.2 from Ch. 6 of Hosford, showing the ratio of the Taylor factor to the reciprocal Schmid factor,  $M/(1/m)$ , for axisymmetric flow with  $\{111\}\langle 110\rangle$  slip. Orientations near 110 exhibit the largest ratios and might therefore be expected to deviate most readily from the Taylor model.

## *Single: Multiple comparison*

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- The result expressed as a ratio of the Taylor factor to the reciprocal Schmid factor is that orientations near 100 or 111 exhibit negligible differences whereas orientations near 110 exhibit the largest differences. This suggests that the latter orientations are the ones that would be expected to deviate most readily from the Taylor model. In wire drawing of bcc metals, this is observed\*: the grains tend to deform in plane strain into flat ribbons. The flat ribbons then curl around each other in order to maintain compatibility.

\*W. F. Hosford, *Trans. Met. Soc. AIME*, 230 (1964), pp. 12.

# Plane Strain Compression

Objective

Properties

Vectorz.

Taylor-  
factor

Sngl.-slip

RC model

- So, for *plane strain* straining paths, the velocity gradient has the form:

$$L = \begin{pmatrix} \pm\delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mp\delta \end{pmatrix}$$

The von Mises equivalent strain for such a tensile strain is always  $2/\sqrt{3}\delta = 1.155\delta$ . Note that this value leads to different results for the Taylor factor, compared to examples in, e.g., Reid and in Bunge (but is consistent with the definitions in the LApp and VPSC codes).

# *Relaxed Constraints*

Objective

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RC model

- An important modification of the Taylor model is the Relaxed Constraints (RC) model. It is important because it improves the agreement between experimental and calculated textures. The model is based on the development of large aspect ratios in grain shape with increasing strain. It makes the assumption that certain components of shear strain generate displacements in volumes that are small enough that they can be neglected. Thus any shear strain developed parallel to the short direction in an elongated grain (e.g. in rolling) produces a negligible volume of overlapping material with a neighboring grain.

# *Relaxed Constraints, contd.*

Objective

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RC model

- In the case of rolling, both the  $\varepsilon_{13}$  and the  $\varepsilon_{23}$  shears produce negligible compatibility problems at the periphery of the grain. Thus the RC model is a relaxation of the strict enforcement of compatibility inherent in the Taylor model.

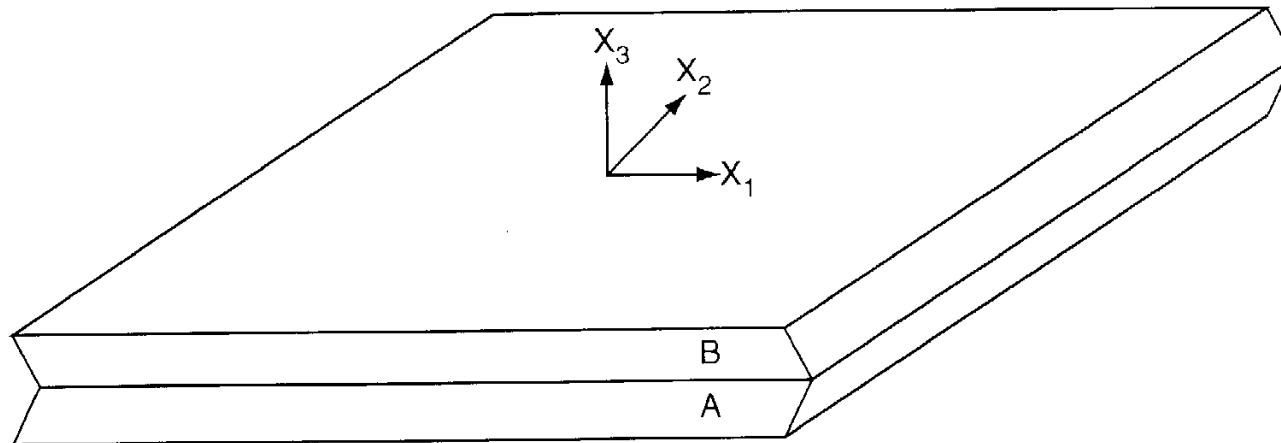


Fig. 21. A flat bicrystal, schematic.

# Relaxed Constraints, contd.

Objective

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RC model

- Full constraints vs. Relaxed constraints

$$D_{11}^C = \sum_s \dot{\gamma}^{(s)} m_{11}^{(s)}$$

$$D_{22}^C = \sum_s \dot{\gamma}^{(s)} m_{22}^{(s)}$$

$$D_{23}^C = \sum_s \dot{\gamma}^{(s)} m_{23}^{(s)}$$

$$D_{31}^C = \sum_s \dot{\gamma}^{(s)} m_{31}^{(s)}$$

$$D_{12}^C = \sum_s \dot{\gamma}^{(s)} m_{12}^{(s)}$$

5 strain BCs

$$D_{11}^C = \sum_s \dot{\gamma}^{(s)} m_{11}^{(s)}$$

$$D_{22}^C = \sum_s \dot{\gamma}^{(s)} m_{22}^{(s)}$$

$$\sigma_{23}^C = 0$$

$$\sigma_{31}^C = 0$$

$$D_{12}^C = \sum_s \dot{\gamma}^{(s)} m_{12}^{(s)}$$

mixed strain &amp; stress BCs

## *RC model (rate sensitive)*

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- With only 3 boundary conditions on the strain rate, the same equation must be satisfied but over fewer components.

$$D^C = \dot{\varepsilon}_0 \sum_s \left| \frac{m^{(s)} : \sigma^c}{\tau^{(s)}} \right|^{n^{(s)}} m^{(s)} \operatorname{sgn}(m^{(s)} : \sigma^c)$$

- Note: the Bishop-Hill maximum work method can still be applied to find the operative stress state: one uses the 3-fold vertices instead of the 6- and 8-fold vertices of the single crystal yield surface.

## *Relaxed Constraints*

Objective

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RC model

- Despite the crude nature of the RC model, experience shows that it results in superior prediction of texture development in both rolling and torsion.
- It is only an approximation! Better models, such as the LAMEL model, the self-consistent model (and finite-element models) account for grain shape more accurately.



# *Effect of RC on texture development*

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RC model

- In (fcc) rolling, the stable orientation approaches the Copper instead of the Taylor/Dillamore position.

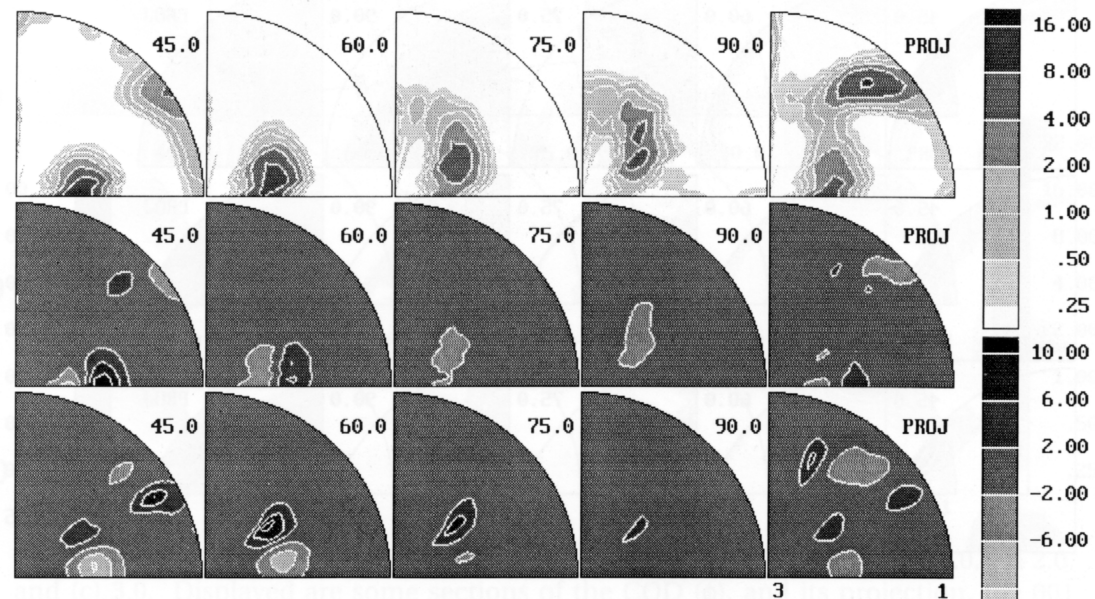


Fig. 4. Simulations as in Fig. 2c, but under full constraints (top row). Difference CODs, against this FC base, under standard RC, as in Fig. 2c (middle row), and under fully relaxed constraints from the beginning (bottom row).

# *Y.S. for textured polycrystal*

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RC model

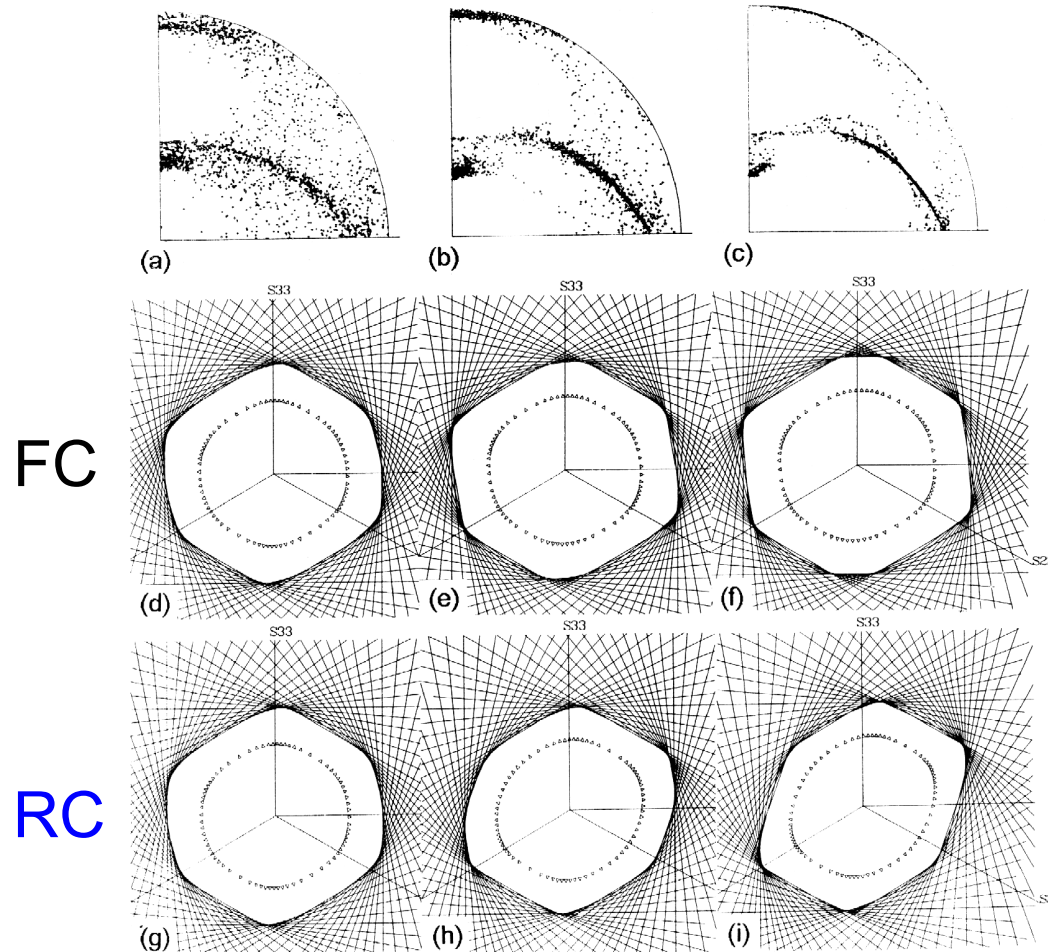


Fig. 15. Simulated rolling textures (using the RC-model) after a von Mises strain of (a) 1.0, (b) 2.0, and (c) 3.0. The corresponding predicted yield surfaces are shown underneath. In (d) through (f), the FC model is used for the property simulation; in (g) through (i), the effect of RC is also accounted for in the yield-surface calculation. The yield surface envelopes refer to a rate sensitivity of  $1/33$ , the inner points to  $1/3$ .

Kocks: Ch.10

Note sharp vertices for strong textures at large strains.

Yield surfaces based on highly elongated grains and the RC model

# Summary

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RC model

- Vectorization of stress, strain tensors.
- Definition, explanation of the Taylor factor.
- Comparison of single and multiple slip.
- Relaxed Constraints model with mixed boundary conditions, allowing for the effect of grain shape on anisotropy.