A note on anisotropic percolation

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Abstract

We consider an anisotropic independent bond percolation model on $\mathbb{Z}_+^2$, i.e. we suppose that the vertical edges of $\mathbb{Z}_+^2$ are open with probability p and closed with probability $1-p$, while the horizontal edges of $\mathbb{Z}_+^2$ are open with probability αp and closed with probability $1-\alpha p$, with $0 < p, \alpha < 1$. Let $x = (x_1, x_2) \in \mathbb{Z}_+^2$, with $x_1 < x_2$, and $x' = (x_2, x_1) \in \mathbb{Z}_+^2$. It is natural to ask how behaves the two point connectivity function $P_{p,\alpha}(\{0 \leftrightarrow x\})$, and whether anisotropy in percolation probabilities implies the strict inequality $P_{p,\alpha}(\{0 \leftrightarrow x\}) > P_{p,\alpha}(\{0 \leftrightarrow x'\})$. In this note we give affirmative answer at least for some regions of the parameters involved.

Keywords: percolation; anisotropy; lattice animals

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§1. *Introduction*

Anisotropic systems are a classical subject in statistical mechanics. In particular, there is a large literature studying anisotropy in fundamental lattice models such as Ising/Potts models and percolation processes. In this paper we focus our attention on anisotropic Bernoulli percolation in the square lattice. The Bernoulli percolation process on the d-dimensional cubic lattice, proposed originally by Broadbent and Hammersley in 1957 to describe the diffusion of water in porous materials, is nowadays used to model a wide number of situations of disordered systems in many areas of physics and applied sciences in general. In particular, the anisotropic version of the two-dimensional Bernoulli percolation is useful in topics of condensed matter and solid state physics, see e.g. recently, [2], [5] and references therein. In spite of its straightforward mathematical formulation, there are very basic questions about 2D anisotropic percolation which have not yet been established rigorously. In this short note we answer partially to one of these questions which is directly related to the anisotropy of the model. In order to describe the problem we need to introduce some notations.

Let $\mathbb{Z}_+^2 = (\mathbb{V}, \mathbb{E})$ be the graph with vertex set $\mathbb{V} = \{x = (x_1, x_2) : x_1 \in \mathbb{Z}_+,\ x_2 \in \mathbb{Z}_+\}$ and edge set $\mathbb{E} = \{e = \{x, y\} : x \in \mathbb{V},\ y \in \mathbb{V},\ |x-y| = 1\}$, where $|x-y| = |x_1 - y_1| + |x_2 - y_2|$ is the usual path distance in $\mathbb{Z}_+^2$. The set $\mathbb{E}$ is naturally partitioned into two disjoint subsets $\mathbb{E}_v$ and $\mathbb{E}_h$. Namely, $\mathbb{E}_h = \{e =$

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$\{(x_1, x_2), (y_1, y_2)\} \in \mathbb{E} : \ x_2 = y_2\}$ and $\mathbb{E}_v = \{e = \{(x_1, x_2), (y_1, y_2)\} \in \mathbb{E} : \ x_1 = y_1\}$. We say that e is a horizontal edge if $e \in \mathbb{E}_h$, while an edge e is called a vertical edge if $e \in \mathbb{E}_v$.

We define a configuration ω of the system as a function $\omega : \mathbb{E} \to \{0,1\} : e \mapsto \omega(e)$. So, given a configuration ω of the system, we say that the edge e is open if $\omega(e) = 1$ while e is closed if $\omega(e) = 0$. We denote by Ω the set of all configurations of the system. We now suppose that each vertical edge $e \in \mathbb{E}_v$ is open with probability p and closed with probability $1-p$, independently from all other edges, and each horizontal edge $e \in \mathbb{E}_h$ is open with probability αp and closed with probability $1 - \alpha p$, independently from all other edges, where $\alpha \in (0, 1]$. We denote by $\mu(e)$ the Bernoulli measure with parameter p if $e \in \mathbb{E}_v$ or αp if $e \in \mathbb{E}_h$.

Then, the anisotropic bond percolation process on $\mathbb{Z}^2_+$ is described by the probability space $(\Omega, \mathcal{F}, P_{p,\alpha})$, where $\Omega = \{0,1\}^{\mathbb{E}}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets in Ω, and $P_{p,\alpha} = \prod_{e \in \mathbb{E}} \mu(e)$ is the product of Bernoulli measures.

Given $x, y \in \mathbb{V}$, let $\{x \leftrightarrow y\}$ be the following event: there is a path γ of open edges connecting x to y. Then its probability $P_{p,\alpha}(\{x \leftrightarrow y\})$ is called the two-point connectivity function of the process. Let now $x = (x_1, x_2) \in \mathbb{V}$, with $x_1 < x_2$, and $x' = (x_2, x_1) \in \mathbb{V}$. If the model is isotropic, i.e. $\alpha = 1$, it is immediate, by symmetry, that $P_{p,\alpha}(\{0 \leftrightarrow x\}) = P_{p,\alpha}(\{0 \leftrightarrow x'\})$ where 0 denotes the origin in $\mathbb{Z}^2_+$. Concerning the anisotropic case, i.e. $\alpha < 1$, in 1997, E. Andjel raised the question whether the strict inequality $P_{p,\alpha}(\{0 \leftrightarrow x\}) > P_{p,\alpha}(\{0 \leftrightarrow x'\})$ is true or not [1]. Crudely, the problem is to prove or disprove the following conjecture:

Conjecture. *Let $x = (x_1, x_2) \in \mathbb{V}$ and $x' = (x_2, x_1) \in \mathbb{V}$ with $x_1 < x_2$, then the strict inequality*

$$P_{p,\alpha}(\{0 \leftrightarrow x\}) > P_{p,\alpha}(\{0 \leftrightarrow x'\}) \tag{1.1}$$

holds for every $x \in \mathbb{V}$, for every $p \in (0,1)$ and for every $\alpha \in (0,1)$.

In spite of its apparent simplicity the question appears to be a stubborn one. We learned recently that P. W. Kasteleyn was asking similar type questions in the middle of eighties. Of course, in the best of all possible worlds, one would like to prove the conjecture without any restriction, but as far as the general problem seems difficult to resolve (if true), one could try to give partial answer to the conjecture in regions that are still interesting from a "physical" point of view.

In this spirit, it is important to understand how the quantity $P_{p,\alpha}(\{0 \leftrightarrow x\})$ is expected to depend on parameters p and α and on the vertex x.

First of all let us consider the dependence of $P_{p,\alpha}(\{0 \leftrightarrow x\})$ from the parameter p. This parameter measures the tendency of the system to have its edges open or closed, no matter of their orientation. Namely, a small p means that the system tends to keep its edges closed while p near 1 means that the system prefers to have its edges open. Intuition does not suggest any particular reason for the inequality (1.1) to depend in a special manner on this parameter. Namely, we don't expect the parameter p particulary important from the physical point of view for the conjecture to be true. So, it would be interesting to show the conjecture at *least for some region of the parameter p.*

The story is completely different for the parameter α. This is the key parameter which measures the degree of anisotropy of the system. Namely α close to 1 means that the system is weakly anisotropic while α near 0 is the opposite situation. Clearly, it should be easier to prove the inequality (1.1) for α near zero than for α near 1. The latter could be the "dangerous" regime, i.e. where the strict inequality (1.1) could not be true.

So one would like to prove the conjecture as uniformly as possible in the parameter α. Namely one would like to prove that, as far as $\alpha < 1$, which is to say as far as *there is* anisotropy in the system, the inequality (1.1) is true at least for some regions of the parameter p.

Finally, let us consider the dependence of the quantity $P_{p,\alpha}(\{0 \leftrightarrow x\}$ from x. $P_{p,\alpha}(\{0 \leftrightarrow x\}$ depends clearly on the distance $|x|$ of vertex x from the origin but also on the slope $\rho = x_2/x_1$ of the vector x with the positive x axis. In particular, if $|x| \to \infty$ and simultaneously $\rho \to 1^+$, i.e. x goes to infinity in such way that the angular distance of x from the diagonal goes to zero, then one should reasonably expect that $P_{p,\alpha}(\{0 \leftrightarrow x\} - P_{p,\alpha}(\{0 \leftrightarrow x'\} \to 0$ for any value of p and α. So it seems very difficult to prove the inequality (1.1) *uniformly* in x, due exactly to the region where $|x| \to \infty$ and $\rho \to 1$. On the other hand these regions does nor seems particulary interesting from the "physical" point of view, since $|x| \to \infty$ and $\rho \to 1$ means that x tends to stay on the diagonal. So, in trying to prove the inequality (1.1) one could demand a uniformity in x in the sense that inequality (1.1) holds for all distances $|x|$ as far as the slope ρ of x is not less than a fixed value greater that 1.

§2. *The main result and some notations*

In this paper we give a partial proof of the Andjel conjecture in the spirit of the discussion of section 1. Namely we are able to show that, if the system is anisotropic, even very weakly anisotropic, and the vertex x is at a fixed positive angular distance from the diagonal, even very small, the inequality (1.1) holds true at least in some regions of the parameter p. Our result can be summarized by the following theorem

Theorem. *Let $x = (x_1, x_2) \in \mathbb{V}$ and $x' = (x_2, x_1) \in \mathbb{V}$ with $x_1 < x_2$ (i.e. $\rho > 1$). Then, for any $\alpha \in (0, 1)$, there exists a $p^*(\alpha, \rho) > 0$ such that*

$$P_{p,\alpha}(\{0 \leftrightarrow x\}) > P_{p,\alpha}(\{0 \leftrightarrow x'\})$$

is true for all $p < p^*$.

The restriction $p < p^*$ (actually p sufficiently small) is due only to technical reasons, so we expect that by using the appropriate techniques in the different regions of p, the validity of the conjecture could be extended for p varying in the whole interval $(0, 1)$ or at least for p in the whole subcritical regime. We recall that, as originally proved in [4] (see also [3]) the subcritical regime for this model occurs when $p(1 + \alpha) < 1$. The conjecture seems to be less relevant from the physical point of view in the supercritical region, since here the connectivity

$P_{p,\alpha}(\{0 \leftrightarrow x\})$ is a less interesting quantity, due to the non zero probability that 0 and x are contained in the infinite cluster.

We also remark that the inequality (1.1) could be proved for larger (but still subcritical) values of p. E.g., via a calculation similar to the one presented here (but easier) it is possible to show that for all $\rho > 1$ and for all α is smaller than a fixed quantity $\alpha^* = \alpha^*(\rho)$ (actually, α sufficiently small), the equality (1.1) holds true for all $p < 1/3$. However this regime is less interesting from a physical point of view, since the system here presents in general not weak anisotropy.

In order to obtain the proof we will need to introduce some notations. In general, if V is any finite set, we denote by $|V|$ the number of elements of V. A lattice animal is defined to be a connected and finite subgraph $A = (V_A, E_A) \subset \mathbb{Z}^d_+$, with vertex set V_A and edge set E_A. We will denote by $\partial_e A$ the edge boundary of a, i.e. the set $\partial_e A = \{e \in \mathbb{E} - E_A : e \cap V_A \neq \emptyset\}$. Given an animal $A = (V_A, E_A)$, we will denote shortly $|A| = |E_A|$ as the number of edges constituting A, we also denote $|A|_h$ as the number of horizontal edges of A, and consistently with previous notations $|\partial_e A|_v$ and $|\partial_e A|_h$ denote the number of vertical edges and horizontal edges respectively in $\partial_e A$, and $|\partial_e A| = |\partial_e A|_v + |\partial_e A|_h$ is the total number of edges in $\partial_e A$. Finally, we will denote by $\mathcal{A}_x$ the set of all lattice animals in $\mathbb{E}$ such that $\{0, x\} \subset V_A$.
A self avoiding path $\gamma \subset \mathbb{Z}^d_+$, joining 0 to x, is an animal $\gamma = (V_\gamma, E_\gamma) \in \mathcal{A}_x$ such that $V_\gamma = \{x^1, \dots, x^n\}$ ($x^i \neq x^j$ if $i \neq j$), with $x^1 = 0$, $x^n = x$, $E_\gamma = \{e_1, \dots, e_{n-1}\}$, where $e_i = \{x^i, x^{i+1}\}$. We denote by $|\gamma|$ the number of edges contained in γ. We also denote by $|\gamma|_h$ the number of horizontal edges contained in γ. Finally Γ_x will denote the set of all self avoiding paths connecting the origin to x. A path $\gamma \in \Gamma_x$ is minimal if $|\gamma| = |x|$. Observe that the number C_x of minimal paths connecting 0 to x is given by

$$C_x = \sum_{\substack{\gamma \in \Gamma_x \\ |\gamma| = |x|}} 1 = \frac{(x_1 + x_2)!}{x_1! x_2!} \tag{2.1}$$

Note also that, for all paths $\gamma \in \Gamma_x$ we have that

$$|\partial_e \gamma| \leq 2|\gamma| + 4 \tag{2.2}$$

§3. *Proof of the Theorem*

In order to get the lower bound for $P_{p,\alpha}(0 \leftrightarrow x)$ we use the standard representation of $P_{p,\alpha}(0 \leftrightarrow x)$ in terms of lattice animals, i.e.

$$P_{p,\alpha}(0 \leftrightarrow x) = \sum_{A \in \mathcal{A}_x} p^{|A|} \alpha^{|A|_h} (1-p)^{|\partial_e A|_v} (1 - \alpha p)^{|\partial_e A|_h}$$

Then, using (2.2) we get

$$P_{p,\alpha}(0 \leftrightarrow x) > \sum_{\gamma \in \Gamma_x} p^{|\gamma|} \alpha^{|\gamma|_h} (1-p)^{|\partial_e \gamma|_v} (1 - \alpha p)^{|\partial_e \gamma|_h} \geq$$

$$\geq \sum_{\gamma\in\Gamma_x} p^{|\gamma|}\alpha^{|\gamma|_h}(1-p)^{|\partial_e\gamma|} > \sum_{\substack{\gamma\in\Gamma_x\\|\gamma|=|x|}} p^{|\gamma|}\alpha^{|\gamma|_h}(1-p)^{|\partial_e\gamma|} >$$

$$> C_x\alpha^{x_1}(1-p)^4\Big[p(1-p)^2\Big]^{x_1+x_2}$$

So, we obtained the lower bound

$$P_{p,\alpha}(0\leftrightarrow x) > C_x\alpha^{x_1}(1-p)^4\Big[p(1-p)^2\Big]^{x_1+x_2} \tag{3.1}$$

Now we will get an upper bound for $P_{p,\alpha}(0\leftrightarrow x')$. First observe that if $x=(0,1)$, then it is easy to check that $P_{p,\alpha}(0\leftrightarrow x) > P_{p,\alpha}(0\leftrightarrow x')$, $\forall\ p,\alpha\in(0,1)$. Indeed,

$$P_{p,\alpha}(0\leftrightarrow x) = P_{p,\alpha}(0\leftrightarrow x\leftrightarrow x') + P_{p,\alpha}(\{0\leftrightarrow x\}\backslash\{0\leftrightarrow x'\}) =$$

$$= P_{p,\alpha}(0\leftrightarrow x\leftrightarrow x') + p(1-\alpha p) > P_{p,\alpha}(0\leftrightarrow x\leftrightarrow x') + \alpha p(1-p) =$$

$$= P_{p,\alpha}(0\leftrightarrow x\leftrightarrow x') + P_{p,\alpha}(\{0\leftrightarrow x'\}\backslash\{0\leftrightarrow x\}) = P_{p,\alpha}(\{0\leftrightarrow x'\}$$

Suppose now $x_1+x_2\geq 2$. In this case

$$P_{p,\alpha}(\{0\leftrightarrow x'\} \leq \sum_{\gamma\in\Gamma_{x'}} p^{|\gamma|}\alpha^{|\gamma|_h} \leq p^{|x|}\alpha^{x_2}\sum_{n=|x|}^{\infty} p^{n-|x|}N(x,n) =$$

$$p^{|x|}\alpha^{x_2}\sum_{n\leq\frac{3}{2}|x|} p^{n-|x|}N(x,n) + p^{|x|}\alpha^{x_2}\sum_{n>\frac{3}{2}|x|} p^{n-|x|}N(x,n) \tag{3.2}$$

where

$$N(x,n) = \sum_{\substack{\gamma\in\Gamma_{x'}\\|\gamma|=n,}} 1$$

is the number of paths of length n connecting 0 to x' which, by symmetry, is equal to the number of paths n connecting 0 to x.
We get an upper bound for the second term in the left hand side of (3.2) using the estimate

$$N(x,n) = \sum_{\substack{\gamma\in\Gamma_{x'}\\|\gamma|=n}} 1 \leq 4\cdot 3^{n-1} \tag{3.3}$$

So

$$p^{|x|}\alpha^{x_2}\sum_{n>\frac{3}{2}|x|} p^{n-|x|}N(x,n) \leq \frac{4}{3}p^{|x|}\alpha^{x_2}\sum_{n>\frac{3}{2}|x|} p^{n-|x|}3^n =$$

$$= \frac{4}{3}p^{|x|}\alpha^{x_2}\sum_{m>\frac{1}{2}|x|} p^m 3^{m+|x|} < \frac{4}{3}p^{|x|}\alpha^{x_2}\sum_{m>\frac{1}{2}|x|} p^m 3^{3m}$$

The last series in the r.h.s. of equation above converges if $p < 1/27$ and in this case we get the upper bound

$$p^{|x|}\alpha^{x_2}\sum_{n>\frac{3}{2}|x|} p^{n-|x|}N(x,n) < \frac{36p^{x_1+x_2+1}\alpha^{x_2}}{1-27p} \quad , \quad \text{for} \;\; p < \frac{1}{27} \tag{3.4}$$

Concerning now the first term in the left hand side of equation (3.2), we use a more careful estimate of the factor $N(x,n)$ when n is small. We follow [6], page 446.

$$p^{|x|}\alpha^{x_2}\sum_{n\leq\frac{3}{2}|x|} p^{n-|x|}N(x,n) = p^{|x|}\alpha^{x_2}\sum_{m\leq\frac{1}{2}|x|} p^{m}N(x,m+|x|)$$

Now observe that

$$N(x,m+|x|) \leq 3^m C_x \binom{m+|x|}{m} = 3^m\, C_x \binom{|x|}{m}\prod_{j=1}^{m}\frac{|x|+j}{|x|-m+j} \leq$$

$$\leq 3^m\, C_x \binom{|x|}{m}\left[\frac{|x|+1}{|x|-m+1}\right]^m \leq 6^m\, C_x \binom{|x|}{m}$$

So

$$p^{|x|}\alpha^{x_2}\sum_{n\leq\frac{3}{2}|x|} p^{n-|x|}N(x,n) \leq C_x p^{|x|}\alpha^{x_2}\sum_{m\leq\frac{1}{2}|x|}\binom{|x|}{m}(6p)^m \leq$$

$$\leq C_x p^{|x|}\alpha^{x_2}\sum_{m=0}^{|x|}\binom{|x|}{m}(6p)^m = C_x p^{|x|}\alpha^{x_2}\,[1+6p]^{|x|}$$

Then, if $x_1 + x_2 \geq 2$ and $p < \frac{1}{27}$, we have the second upper bound:

$$P_{p,\alpha}(\{0 \leftrightarrow x'\} \leq p^{x_1+x_2}\alpha^{x_2}\left[\frac{36p}{1-27p} + C_x\,(1+6p)^{x_1+x_2}\right].$$

Comparing this upper bound with the lower bound previously obtained, we have that $P_{p,\alpha}(0 \leftrightarrow x) > P_{p,\alpha}(0 \leftrightarrow x')$ if

$$\alpha^{x_2-x_1}\left[\frac{36p}{1-27p} + C_x\,(1+6p)^{x_1+x_2}\right] < \; C_x(1-p)^4\Big[(1-p)^2\Big]^{x_1+x_2}$$

Inequality (1.1) above is satisfied for all α such that

$$\alpha < \tilde{\alpha}(p, x_1, x_2)$$

where the function $\tilde{\alpha}(p, x_1, x_2)$ is defined for all $p < 1/27$, all $x_1 + x_2 \geq 2$ and $x_2 - x_1 \geq 1$ and it is given explicitly by

$$\tilde{\alpha}(p, x_1, x_2) = \left[\frac{C_x(1-p)^4\Big[(1-p)^2\Big]^{x_1+x_2}}{\frac{36p}{1-27p} + C_x\,(1+6p)^{x_1+x_2}}\right]^{1/(x_2-x_1)}$$

Note now that $\tilde{\alpha}(p, x_1, x_2) \to 1^-$ as $p \to 0^+$. Observe also that, for fixed slope $\rho > 1$, $\lim_{|x|\to\infty} \tilde{\alpha}(p, x_1, x_2) = f(p, \rho)$ and $\lim_{p\to 0^+} f(p, \rho) = 1$. This ends the proof of the theorem.

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References

[1] Andjel E.: *Private communication* (1997)

[2] Celzard, A.; Mareche, J. F.: *Non-universal conductivity critical exponents in anisotropic percolating media: a new interpretation* Physica A, **317** (3-4), 305-312 (2003)

[3] Grimmett, G.: *Percolation*, second edition, Springer Verlarg, New York (1999).

[4] Kesten, H.: *Percolation Theory for Mathematicians*. Birkhäuser. Boston (1982).

[5] Nguyen, V.; Dang, D. T.: *Variable range hopping in finite one-dimensional and anisotropic two-dimensional systems* Phisica B, **334** (1-2), 88-97 (2003)

[6] Simon, B. : *The Statistical Mechanics of the lattice gases, vol. I.* Princeton University Press (1993).