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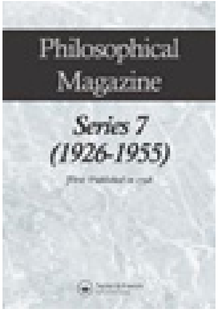

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XLVI. *A Theory of the Plastic Distortion of a Polycrystalline Aggregate under Combined Stresses.*

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Summary.

A general relationship between stress and plastic strain in a polycrystalline aggregate is derived for any metal in which individual crystals deform by slipping over preferred planes under a critical shear stress. Full account is taken of the non-uniform distortion due to mutual constraints between the grains of an aggregate. It is shown that a plastic potential exists which is identical with the yield function. Upper and lower bounds are obtained for an approximate calculation of this function for any applied system of combined stresses.

§1. Introduction.

The macroscopic theory of plastic deformation in a polycrystalline metal is based on observations of the behaviour of the metal in bulk. It rests also on the simplifying hypothesis that the material is locally homogeneous (though not necessarily isotropic). The theory so constructed is found to be adequate as a first approximation when applied to many problems in engineering science and metal technology. There remains, however, the task of relating the macroscopic observations to ones more fundamental, for example, the mechanisms by which a single crystal deforms under load. Such an investigation might hope to attain two main objectives: first, to show which of the several microscopic modes of distortion are chiefly responsible for the behaviour of the aggregate as a whole; and second, to indicate, more precisely than experiment at present can, what refinements should be added to the macroscopic theory and what these might be for any particular metal. Several previous writers have dealt with related problems, and the most important papers will be discussed briefly.

In single crystals of many metals it is known that the main mechanism of plastic distortion on a microscopic (though not atomic) scale is simple shear parallel to preferred planes and directions. At ordinary temperatures both the planes and directions are generally those of closest atomic packing. It is observed that pronounced slip (or glide) is initiated along the particular plane and direction for which the corresponding component of shearing stress first reaches a certain critical value under increasing external load. This critical value depends on the temperature and rate

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* Communicated by the Authors.

of straining, but not on the crystal orientation or the type of load (provided the loading-path is not such as to induce hysteresis effects). This has been established for metals such as copper, aluminium, nickel, magnesium, zinc and cadmium by the experiments of Mark, Polanyi and Schmid (1922), Taylor and Elam (1923), Polanyi and Schmid (1923) and Schmid (1924), among others. Double slipping is observed if the critical shear stress is reached simultaneously on two of the possible planes and directions.

The problem of predicting the tensile yield stress of a polycrystalline face-centred cubic metal was considered by Sachs (1928) and by Cox and Sopwith (1937) on the assumptions that (i) each grain is subjected to a uniaxial stress parallel to the specimen axis sufficient to initiate slipping in the most highly stressed direction ; (ii) all orientations are equally likely ; (iii) each grain is in the same state of work hardening. The resultant axial load on the specimen was calculated as the sum of the individual loads in the constituent grains. The value obtained for the macroscopic uniaxial yield stress was $2{\cdot}2\tau$ (approximately), where τ is the shear yield stress of a single crystal. This treatment is open to the objections that the grains could not form a coherent whole if only a single homogeneous glide occurred in each, and that the necessary conditions of equilibrium could not be satisfied across grain boundaries if the stress in each were a simple tension of varying amount.

Kochendörfer (1941) added the further hypothesis that the grains extend by the same amounts in the direction of the specimen axis. This enabled him to calculate the stress-strain curve of the aggregate. For in each grain

$$\sigma/\tau = d\gamma/d\epsilon = m$$

where σ is the axial stress in a grain, $d\epsilon$ is the common axial strain-increment, $d\gamma$ is the increment of shear strain in the slip direction, and m is a dimensionless factor depending only on orientation. (In passing, it may be noted that the virtual work equation $\sigma d\epsilon = \tau d\gamma$ is naturally satisfied). We now interpret the various symbols as mean values over all orientations, so that $m \simeq 2{\cdot}2$ from Sachs' calculations. It follows that the stress-strain curve of the aggregate is

$$\sigma = m\tau(\gamma) = m\tau(m\epsilon)$$

where $\tau(\gamma)$ is the shear-hardening curve (the transition from $d\gamma = m d\epsilon$ to $\gamma = m\epsilon$ is legitimate since m is independent of the amount of strain on the assumption of continuing isotropy). However, the best fit of this $\sigma(\epsilon)$ relation to Taylor's experimental curve for aluminium gives a value of about $3{\cdot}1$ for m.

In a recent paper (1950) Calnan and Clews considered the particular directions in which a pure tension would need to be applied to operate simultaneously either 4, 6, or 8 shears in a face-centred cubic lattice. They assumed the stress state in each grain to be a uniaxial tension of this kind and of the requisite amount. Each grain is assumed to undergo the specimen extension (and not the strain as a whole). Thus, the objections raised against previous work with regard to the independent action of the grains apply here, also. Despite this Calnan and Clews' computations led to a stress-strain curve in fair agreement with that measured for polycrystalline aluminium.

A more realistic calculation of the tensile yield stress of a face-centred cubic aggregate is due to Taylor (1938), who assumed that each grain undergoes the same uniform strain (so maintaining cohesion). This requires, in general, the operation of at least 5 independent shears out of the 12 operable at ordinary temperatures. Hence, if only geometry is considered, many choices are possible for a given strain. Taylor introduced the hypothesis that the actual active set of shears is that for which the sum of their separate magnitudes is least. The hypothesis has no obvious *a priori* justification, but Taylor based it on observations of single crystals under uniaxial stress and on a postulated analogy with the dynamics of a non-conservative mechanical system. By adding the further assumption that the shear hardening of a crystal during simultaneous glide depends only on the sum of the absolute shears, Taylor was able to compute a stress-strain curve for an aluminium aggregate in fairly close agreement with the measured curve. There are two main criticisms of this approach: first, it is not proved that a combined stress could always be found to operate any geometrically possible set of shears (without exceeding the critical shear stress in non-active directions); and, second, no account is taken of stress-continuity conditions across grain boundaries.

All these previous investigations have been confined to an aggregate under simple tension or compression. In the present paper no such restriction is placed on the applied load.

§2. THE MACROSCOPIC THEORY OF PLASTICITY.

In order to appreciate the aim of the present work it is necessary to be familiar with the premises of the macroscopic theory of plasticity. These will be summarized here; a full discussion has been given by Hill (1950).

The theory assumes a locally homogeneous material (possibly anisotropic) having a sharp yield point after work-hardening. The yield criterion for a metal in a given state of hardening is assumed not to depend on the hydrostatic component σ of the applied stress σ_{ij}, and is usually taken to have the form

$$f(\sigma'_{ij})=c\,;\quad \sigma'_{ij}=\sigma_{ij}-\sigma\delta_{ij}. \qquad \dots\dots\dots \quad (1)$$

This equation may be regarded as defining a surface in stress hyperspace; the surface is " cylindrical ", with generators parallel to the direction δ_{ij}. An assumption adequate for many applications, though not necessary in principle, is that the function f does not involve parameters depending on the previous strain-history. The dependence of the shape and size of the yield surface on strain-history is thus taken to be expressible through the single parameter c. Infinitesimal changes of stress for which $df=0$ maintain the element on the point of yielding, and cause only elastic changes of strain in a hardening material. The sign of f is conventionally chosen so that $df>0$ for an increment of stress producing further plastic strain, while if $df<0$ the element unloads elastically.

The elastic part of the strain is neglected henceforward. The relations between stress and strain are taken to be

$$d\epsilon_{ij}=h\frac{\partial g}{\partial \sigma_{ij}}df\,;\ df\geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

g (the plastic potential) and h are scalar functions of the reduced stress σ'_{ij}; parameters in g depending on strain-history are held constant in the differentiation. When the element is isotropic f, g and h must be functions only of the two independent invariants of the tensor σ'_{ij}. Since g does not involve the hydrostatic part of the stress, (2) implies that there is no plastic volume change.

It is normally assumed, in the absence of definite experimental evidence, that the yield function f and the plastic potential g are the same*. With this simplification equation (2) becomes

$$d\epsilon_{ij}=h\frac{\partial f}{\partial \sigma_{ij}}df\,;\ df\geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Certain extremum principles (and an associated uniqueness theorem) may then be proved (Hill 1950). The proof of one of them will be repeated here for the sake of a later analogy.

(i) *Maximum Work Principle.*

Equation (2) may be interpreted as asserting that the strain increment (represented in stress hyperspace) is parallel to the corresponding outward normal to the yield surface. In order that a unique state of reduced plastic stress should correspond to a prescribed increment of strain it is evidently necessary for this surface to enclose, and be concave at all points to, the origin.

There will be two generators on the surface where the normals are parallel to a given direction; uniqueness is secured by the condition that the plastic work is positive.

Suppose, now, that an element is in a given state of hardening, defined by a certain yield-surface. Consider a plastic stress state σ_{ij} and corresponding strain-increment $d\epsilon_{ij}$. If σ^*_{ij} is a stress state lying within or upon the yield surface, the vector $\sigma_{ij}-\sigma^*_{ij}$ is inclined at an acute angle to the outward normal at σ_{ij} (the yield surface being cylindrical and concave to the origin). Hence (using the summation convention)

$$(\sigma_{ij}-\sigma^*_{ij})d\epsilon_{ij}\geqslant 0, \quad d\epsilon_{ij}\neq 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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* More generally, g is taken equal to any convenient multiple or power of f. The resulting stress–strain relations are always the same if h is suitably altered.

the equality holding only when σ_{ij} and σ^*_{ij} differ by a hydrostatic component. This has the interpretation that the work done by the actual stress in a prescribed strain-increment is greater than that done by any other stress not violating the yield condition. The result does not involve any assumption about the dependence of the yield surface on the strain-history.

Consider, next, a finite mass undergoing plastic deformation at *all* points. Let σ_{ij} be the actual stress, satisfying the yield condition and the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j}=0, \qquad \ldots\ldots\ldots\ldots \quad (5)$$

at all points. The associated strain-increment must be derived from a continuous displacement increment du_i, such that

$$d\epsilon_{ij}=\frac{1}{2}\left\{\frac{\partial}{\partial x_j}(du_i)+\frac{\partial}{\partial x_i}(du_j)\right\}.$$

If σ^*_{ij} is any stress state satisfying (5) and lying within or on the yield surface for each point of the mass, we have the virtual work equation

$$\int\{(\sigma_{ij}-\sigma^*_{ij})d\epsilon_{ij}\}\,dV=\int\{(F_i-F^*_i)du_i\}\,dS,$$

where the first integral is taken through the volume, and the second over the surface of the mass, F_i and F^*_i being the ith components of the respective outward forces acting on unit surface area. Hence, from (4)

$$\int\{(F_i-F^*_i)\,du_i\}\,dS\geqslant 0, \qquad \ldots\ldots\ldots \quad (6)$$

the equality holding only when the stress states differ by a uniform hydrostatic stress. Equation (6) is true whether or not the states of hardening and anisotropy are non-uniform, and no assumption is made as to how they depend on strain-history.

(ii) *Converse of Maximum Work Principle.*

Suppose it to be given that, for any prescribed strain-increment, the corresponding state of plastic stress in an element is such that the work done is stationary with respect to infinitesimally near stress states, lying on the yield surface. Then it is evident that the normal to the yield surface at the corresponding point must be parallel to the strain-increment vector. A plastic potential therefore exists and it is identical with the yield function. The latter, moreover, must be cylindrical with generators parallel to the direction δ_{ij}, since any possible strain-increment has zero hydrostatic part (there being no plastic change of volume). If the work done is an absolute maximum, with respect to *all* stress states within or on the yield surface, the surface must be concave to the origin at every point.

This is the basis of the method adopted later to derive equation (3) from the properties of single crystals.

(iii) *Complementary Minimum Principle.*

Consider a mass in an equilibrium plastic state $(\sigma_{ij}, d\epsilon_{ij})$ and let $d\epsilon^*_{ij}$ be any strain-increment; $d\epsilon_{ij}$ and $d\epsilon^*_{ij}$ are derived from continuous displacement increments du_i and du^*_i taking the same values on the surface and having zero divergence everywhere. It may be proved (Hill, *op. cit.*, p. 67) that

$$\int(|\sigma'_{ij}||d\epsilon_{ij}|)\,dV \leqslant \int|(\sigma'_{ij}||d\epsilon^*_{ij}|)\,dV, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $$|\sigma'_{ij}| = \sqrt{(\sigma'_{ij}\sigma'_{ij})}, \text{ etc.},$$

provided the material is isotropic and the plastic potential is a *circular* cylinder in principal-stress space; *i.e.* $f = \sigma'_{ij}\,\sigma'_{ij}$ in (3). If the state of hardening is uniform, (7) reduces to Markov's result

$$\int|d\epsilon_{ij}|\,dV \leqslant \int|d\epsilon^*_{ij}|\,dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The analogue of (7) for any plastic potential is

$$\int(\sigma_{ij}\,d\epsilon_{ij})\,dV = \int(\sigma_{ij}\,d\epsilon^*_{ij})\,dV \leqslant \int(\sigma^*_{ij}\,d\epsilon^*_{ij})\,dV,$$

where (4) is used to derive the inequality, σ^*_{ij} being the plastic stress-state corresponding to $d\epsilon^*_{ij}$.

§ 3. Plasticity of a Single Crystal.

We assume that the only mechanism of plastic distortion in a single crystal is by glide parallel to preferred planes and directions, and we examine the consequences of such an assumption. Let there be n possible slip directions $a_1, \ldots a_n$. (on associated glide planes), for a certain crystal lattice. The displacement of any point of the crystal, due to simultaneous infinitesimal shears $d\gamma_1, \ldots d\gamma_n$ (positive or negative), in the respective directions may be found by simple geometry. The corresponding tensor representing the homogeneous strain has five independent components (its hydrostatic part being necessarily zero), each of which is found as a certain linear combination of $d\gamma_1, \ldots d\gamma_n$. For given $d\gamma_k$ $(k=1, \ldots n)$ the strain tensor is thereby uniquely determined.

On the other hand, when the components of the strain tensor are given (the hydrostatic part being zero), we have n unknown $d\gamma_k$ and 5 equations between them. If n is less than 5 (as in a hexagonal metal) a combination of shears cannot be found to produce an *arbitrary* strain. If n is equal to 5 there is a unique set of the $d\gamma_k$ if the determinant of the coefficients is non-zero. If n exceeds 5, a set of 5 slip directions can be selected in any one of ${}^nC_5$ ways, but the corresponding set of shears can only be found when the associated determinant is non-zero, *i.e.* when the slip directions form an *independent* set. It may also be possible to find combinations of 6 or more shears to produce the prescribed strain.

In a fully annealed crystal the critical shear stress appears not to depend on which slip-direction is activated, nor on the sense of the shear. During continued deformation active and non-active directions harden equally, to a first approximation (Taylor and Elam 1925). Small differences are observed, however; for example, double slipping does not always begin at the stress which would be expected if the previously non-active direction hardened at the same rate as the active one. Differences between the critical shear stresses may also be due to microscopic internal stresses set up during cold-working. These stresses can be expected to favour slip in certain directions and senses, and they will therefore be effective whenever the load is varied in orientation or sense. At present there is no clear evidence as to the importance of such effects in crystals of *pure* metals, but a Bauschinger effect has been observed in cold-worked brass, from which it could be largely removed by a mild annealing not reducing the average hardening (Sachs and Shoji 1927).

In the subsequent analysis we allow for these possibilities by associating two critical shear stresses with each of the preferred directions: τ_k for slip in the sense a_k, and τ_k' for slip in the sense $-a_k$. It is assumed that the hydrostatic part of the applied stress is without influence on these critical values; this has been found true in many pure metals (as distinct from alloys) provided the hydrostatic stress is not more than a few times the shear yield stress (*e.g.* Polanyi and Schmid 1923, Taylor 1927).

(i) *Maximum Work Principle for a Single Crystal.*

Let $d\epsilon_{ij}$ be a prescribed increment of strain in a single crystal, and let σ_{ij} be a stress (not necessarily unique) which will produce this strain. Let σ_{ij}^* be any other stress which does not violate the yield conditions. Then

$$dW-dW^*=(\sigma_{ij}-\sigma_{ij}^*)\,d\epsilon_{ij}=\Sigma(\tau-\tau^*)\,d\gamma,$$

where the summation is over all shears of a set equivalent to $d\epsilon_{ij}$ and activated by σ_{ij}. We take each $d\gamma$ to be the shear (positive or negative) in the sense a of the corresponding slip-direction, so that τ and τ^* are then the shear stresses in the same sense. Then, when $d\gamma>0$ in the *kth* direction,

$$\tau-\tau^*=\tau_k-\tau^*\geqslant 0$$

and when $d\gamma<0$ in the *kth* direction,

$$\tau-\tau^*=-\tau_k'-\tau^*\leqslant 0$$

since, by hypothesis $\qquad -\tau_k'\leqslant\tau^*\leqslant\tau_k.$

All products $(\tau-\tau^*)d\gamma$ are therefore positive or zero, and so

$$(\sigma_{ij}-\sigma_{ij}^*)\,d\epsilon_{ij}\geqslant 0. \qquad \text{. . . . . (9)}$$

The equality holds only when τ and τ^* are equal in all active directions, and, if there are at least 5 of these, σ_{ij} and σ_{ij}^* will generally differ only by a hydrostatic stress. Particular strains may be obtainable by applying any one of a number of stresses (not differing merely by a hydrostatic component), but these stresses all do the same work. For if σ^*_{ij} is also a physically possible stress,

$$(\sigma^*_{ij}-\sigma_{ij})\, d\epsilon_{ij} \geqslant 0,$$

in addition to (9), and therefore $\sigma^*_{ij} d\epsilon_{ij}=\sigma_{ij}\, d\epsilon_{ij}$.

It is also to be remarked that many sets of shears (even associated with the same stress) may produce, in certain cases, the same strain. That is, more than one set of shears may be physically, as well as geometrically, possible. However, since there are only 5 disposable stress components it is clear that only for certain sets of 5 slip-directions is it possible to find a stress to activate them *in prescribed senses*, since the critical shear stress must also not be exceeded in the non-active directions.

(ii) *Minimum Shear Principle for a Single Crystal.*

Let $d\epsilon_{ij}$ be a prescribed increment of strain and let σ_{ij} be a stress which will produce this strain by activating a set of shears $d\gamma$. Let $d\gamma^*$ be any set of shears which are geometrically equivalent to the prescribed strain, but which are not necessarily operable by any stress satisfying the yield conditions. The virtual work equation can be written in two ways :

$$\sigma_{ij}\, d\epsilon_{ij}=\Sigma\tau\, d\gamma=\Sigma\tau\, d\gamma^*.$$

If $\tau_k=\tau'_k$ for each direction (no Bauschinger effect),

$$\Sigma\tau\, d\gamma=\Sigma\tau_k\, |d\gamma|, \quad \text{and} \quad \Sigma\tau\, d\gamma^* \leqslant \Sigma\tau_k\, |d\gamma^*|.$$

Hence

$$\Sigma\tau_k\, |d\gamma| \leqslant \Sigma\tau_k\, |d\gamma^*|. \quad \ldots\ldots \quad (10)$$

In particular, when all directions are equally hardened,

$$\Sigma\, |d\gamma| \leqslant \Sigma\, |d\gamma^*|. \quad \ldots\ldots \quad (11)$$

This states that the sum of the absolute values of a physically possible set of shears producing a given strain is less than that for a set which is only geometrically possible. If more than one set is physically possible the sums of their absolute values are equal.

We have thereby proved the hypothesis (11) suggested by Taylor (1938), and extended it (10) to the case when the hardening is unequal.

§ 4. POLYCRYSTALLINE AGGREGATE.

The experimental laws of plastic deformation in an aggregate (as a whole) express relations between macroscopic measures of stress and strain. An ideal experiment is designed so that the measuring device gives an average value of stress or strain over a large number of crystal grains. In the interpretation of such measurements the following two assumptions are involved.

(*a*) The measuring device extends over a volume of such size that the distribution of orientations and hardnesses of crystals (or perhaps fragmented " crystallites ") does not vary significantly from one part to another. That is, the specimen is regarded as homogeneous in the macroscopic sense. This does not preclude a macroscopic state of anisotropy, since the actual orientations are not necessarily randomly distributed over all possible ones.

It is convenient to refer to the smallest cubical volume possessing this property as a " unit " cube.

(*b*) No correlation exists between microscopic stress and position over any plane section of " unit " area. This is the necessary and sufficient condition for the stress resultant over such a unit section to be a single force through the centroid of the section. The Cartesian components of the resultant force per unit area are, of course, the conventional measures of the corresponding components of macroscopic stress. These form a tensor (as a direct consequence of the conditions for linear equilibrium) and the tensor is symmetric if the stress resultant on a plane is a force, and not a force and irreducible couple. If a correlation did exist between microscopic stress and position (and this appears possible in principle), the macroscopic stress tensor would not necessarily be symmetric, and the conditions for angular equilibrium would take another form. These remarks are equally pertinent, of course, to the concept of *microscopic* stress in a single crystal and its dependence on the distribution of interatomic forces.

The definition of a normal component of an infinitesimal *homogeneous* strain is the increment of distance between two points initially unit distance apart in the direction under consideration. The definition of a component of shearing strain is based on the total relative tangential displacements of two pairs of perpendicular planes at unit distance apart. The analogous definition of the macroscopic strain-increment components for a " unit cube " of an aggregate is

$$dE_{ij}=\tfrac{1}{2}\int(l_i\,du_j+l_j\,du_i)\,dS, \quad \ldots\ldots \quad (12)$$

where the integral is taken over the surface of the unit cube, l_i being the unit outward normal, and du_i the incremental microscopic displacement (the coordinate axes are taken parallel to the cube edges). When the microscopic displacement is a continuous function of position (no opening of cavities or sliding of one grain over another), Green's theorem gives

$$dE_{ij}=\frac{1}{2}\int\left\{\frac{\partial}{\partial x_i}(du_j)+\frac{\partial}{\partial x_j}(du_i)\right\}dV,$$

$$=\int(d\epsilon_{ij})\,dV,$$

taken through the volume of the unit cube, where $d\epsilon_{ij}$ is the microscopic strain-increment. Since the latter is a symmetric tensor with zero hydrostatic part so also is dE_{ij}.

Consider the work of deformation within a unit cube. It is

$$dW = \int(\sigma_{ij}\, d\epsilon_{ij})\, dV = \int(\sigma_{ij}\, du_i l_j)\, dS, \quad . \; . \; . \; . \quad (13)$$

provided the displacement function is continuous and the equilibrium equations are satisfied. At grain boundaries, or elsewhere, the microscopic stress is not necessarily continuous in all its components, and only the stress resultants acting on opposite sides of such a surface must be the same for equilibrium.

We now postulate that the grains are distributed in the unit cube in such a way that there is no correlation between any component of the microscopic stress and any component of the displacement over any plane section of unit area. This is so provided that

$$\int\sigma_{ij}\, dA \times \int du_k\, dA = \int(\sigma_{ij}\, du_k)\, dA (i, j, k = 1, 2, 3), \quad . \; . \quad (14)$$

where the integration extends over the unit section. In particular, this is satisfied when the stress is uniform or when the strain is uniform (the stress being distributed according to (*b*) above). Applying (14) to each face of the unit cube, we find from (13) that

$$dW = \int(\sigma_{ij}\, d\epsilon_{ij})\, dV = S_{ij}\, dE_{ij}, \quad . \; . \; . \; . \; . \quad (15)$$

where S_{ij} is the macroscopic stress tensor defined in (*b*).

(i) *Maximum Work Principle for an Aggregate.*

Let S^*_{ij} be any macroscopic stress corresponding to an equilibrium microscopic distribution σ^*_{ij} not violating the yield conditions at any point of a unit cube. Then, from (15),

$$(S_{ij} - S^*_{ij})\, dE_{ij} = \int\{(\sigma_{ij} - \sigma^*_{ij})\, d\epsilon_{ij}\}\, dV \geqslant 0, \quad . \; . \; . \; . \quad (16)$$

since (9) holds at every point. We have thereby derived (4) from assumed properties of a single crystal. It is worth noting that (16) is still true even if du_i is tangentially discontinuous across certain surfaces provided the corresponding frictional stress vanishes. This is approximately true in high-temperature creep where relative slip occurs between grains.

(ii) *Minimum Shear Principle for an Aggregate.*

Let du_i and du^*_i be two continuous displacement distributions, with zero divergence, taking the same values on the surface of a unit cube. du_i is associated with an equilibrium distribution of stress σ_{ij} satisfying the yield conditions. Then

$$\int(\sigma_{ij}\, d\epsilon_{ij})\, dV = \int(\sigma_{ij}\, d\epsilon^*_{ij})\, dV \quad \text{or} \quad S_{ij}\, dE_{ij} = S_{ij}\, dE^*_{ij},$$

and so

$$\int(\Sigma\tau\, d\gamma)\, dV = \int(\Sigma\tau\, d\gamma^*)\, dV.$$

If $\tau_k = \tau'_k$ for each direction at any point (no Bauschinger effect),

$$\Sigma\tau\, d\gamma = \Sigma\tau_k\, |d\gamma|, \quad \Sigma\tau\, d\gamma^* \leqslant \Sigma\tau_k\, |d\gamma^*|.$$

Hence

$$S_{ij}\, dE_{ij} = \int(\Sigma\tau_k\, |d\gamma|)\, dV \leqslant \int(\Sigma\tau_k\, |d\gamma^*|)\, dV. \quad . \; . \; . \quad (17)$$

When the critical shear stress is uniform throughout the aggregate :

$$\int(\Sigma|d\gamma|)\,dV \leqslant \int(\Sigma|d\gamma^*|)\,dV. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This is true whether or not the aggregate has the properties ascribed to the unit cube. There is a close formal analogy between (17) and (7), and between (18) and (8).

On combining (16) and (17) we have for a unit cube, when $\tau_k = \tau_k'$

$$S_{ij}^*\,dE_{ij} \leqslant S_{ij}\,dE_{ij} \leqslant \int(\Sigma\tau_k|d\gamma^*|)\,dV. \quad . \quad . \quad . \quad . \quad (19)$$

Another inequality that will be used later is

$$S_{ij}\,dE_{ij}^* \leqslant \int(\Sigma\tau_k|d\gamma^*|)\,dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where du_i^* is *any* continuous distribution.

It is to be noticed carefully that in establishing these extremum principles for an aggregate there is an implicit assumption, not merely that slip is the only mechanism of distortion, but that it is also a *sufficient* mechanism. In other words, it is assumed that the equations of equilibrium (5) can be satisfied throughout an aggregate by a stress distribution which, at the same time, will everywhere operate sufficient shears to produce a continuous displacement. For this to be mathematically possible, it may well be that the microscopic stress and strain must be allowed to vary *continuously*, and not be restricted to take constant values within each of a finite number of regions (which may be grains or parts of grains). If a continuous variation is envisaged, the relations between microscopic stress and strain, established for a crystal of a certain finite size, must be assumed to apply *at a point*. This, of course, is a procedure that is adopted in every branch of mechanics.

§ 5. CALCULATION OF THE YIELD SURFACE.

In § 4 (i) a maximum work principle for an aggregate has been shown to hold when slipping under a critical shear stress is the only microscopic mechanism of distortion. It follows from § 2 (ii) that a plastic potential governs the relation between macroscopic stress and strain-increment, and that the plastic potential is identical with the yield function. If, therefore, the yield function can be calculated for any particular metal the relationships between the ratios of the stress components and the ratios of the strain-increment components follow immediately from (3).

We consider first what general properties of the yield function follow from the assumption of deformation by slip. It is evident that the hydrostatic part of the applied stress has no influence on yielding ; that is, the yield surface is cylindrical (though not, of course, necessarily circular). This follows since the superposition of a uniform hydrostatic stress throughout a plastic aggregate does not disturb the equilibrium of the microscopic stresses, nor does it alter the active slip-directions at any point, when the influence of elasticity is disregarded. In reality, however, a certain effect of hydrostatic stress would be expected in an aggregate of elastically anisotropic grains.

If the critical shear stress does not depend on the sense of slip, and if the aggregate is free from internal stress when the applied loads are removed after cold-work, the yield surface for that state is symmetrical about the origin. That is, if S_{ij} produces yielding when the aggregate is reloaded under constant stress-ratios so also will $-S_{ij}$. For the elastic compatibility equations are linear in the stresses, and hence if a microscopic distribution σ_{ij} corresponding to S_{ij} can be established by monotonic loading from a stress-free state so also can a distribution $-\sigma_{ij}$ corresponding to $-S_{ij}$. Furthermore, if σ_{ij} satisfies the yield conditions at any point so does $-\sigma_{ij}$, and if $d\epsilon_{ij}$ is the increment of strain associated with σ_{ij}, $-d\epsilon_{ij}$ is the increment associated with $-\sigma_{ij}$. Plastic deformation would therefore be initiated under the stress $-S_{ij}$.

When the aggregate is macroscopically isotropic the yield surface can be shown to possess a six-fold symmetry in principal-stress space (Hill 1950, p. 18). Since this symmetry does not depend on any particular mechanism of plastic distortion, we do not need to discuss it here.

The exact calculation of the yield surface appears to be a matter of some difficulty. In the present paper we shall merely show that it is possible to calculate, with comparative ease, two cylindrical surfaces between which the yield surface must certainly lie when there is no Bauschinger effect. For this purpose we use the inequalities established in the last section. In one case we consider an aggregate in which the stress is uniform but displacement continuity is violated, and in the other case we consider an aggregate in which the strain is uniform but equilibrium is violated.

Take first the uniform stress distribution

$$\sigma^*_{ij}=\lambda^* r_{ij}, \quad r_{ij}=S_{ij}/(S_{ij}S_{ij})^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where, for given stress-ratios r_{ij}, λ^* is to be chosen so that the critical shear stress corresponding to σ^*_{ij} is just attained in one slip-direction at the " weakest " point of the aggregate. It is then certain that σ^*_{ij} does not violate the yield conditions anywhere in the aggregate. σ^*_{ij} is therefore an equilibrium distribution such as is envisaged in (16) (the displacement function being continuous) which leads to

$$(S_{ij}S_{ij})^{1/2}=|S_{ij}|\geqslant\lambda^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Hence, for each direction r_{ij} in stress hyperspace, (22) supplies a lower limit to the length of the corresponding " radius " to the yield surface. Since we know that the surface is cylindrical we need only take stresses S_{ij} with zero hydrostatic part in (21).

Next take any *uniform* strain distribution $d\epsilon^*_{ij}=dE^*_{ij}$ with zero hydrostatic part, and such that $r_{ij}\,dE^*_{ij}>0$. Let $d\gamma^*$ be any of the sets of shears which are equivalent to the strain dE^*_{ij}. $d\gamma^*$ is a function of position since the lattice orientation varies through the aggregate. Let a quantity μ^* be defined such that

$$\mu^*=\int(\Sigma\tau_k\,|d\gamma^*|)\,dV/r_{ij}\,dE^*_{ij}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where r_{ij} is defined in (21) and the integration extends through the unit cube. Then, by (20),

$$|S_{ij}| \leqslant \mu^{*}. \qquad (24)$$

According to (10) the least possible value of μ^{*} for any choice of dE^{*}_{ij} is to be obtained by taking $d\gamma^{*}$ to be a *physically* possible set of shears for each orientation, the " crystallites " being supposed completely independent. Then

$$\mu^{*}=(dE^{*}_{ij}\int\sigma^{*}_{ij}\,dV)/(r_{ij}\,dE^{*}_{ij}), \qquad (25)$$

where σ^{*}_{ij} is a stress which would produce the strain dE^{*}_{ij} in a free crystal. An advantageous choice of dE^{*}_{ij} appears to be $dE^{*}_{ij}=r_{ij}$ (an arbitrary scale factor can obviously be omitted for convenience). Then, since $r_{ij}r_{ij}=1$,

$$\mu^{*}=r_{ij}\int\sigma^{*}_{ij}\,dV. \qquad (26)$$

This value of μ^{*} is equal to the work which would be done if a unit cube of aggregate were split into its constituent regions of uniform orientation and each were separately given the strain r_{ij}.

§ 6. DEPENDENCE OF YIELD SURFACE ON STRAIN-HISTORY.

We have not yet considered how the shape and size of the yield surface depend on the previous strain-history. For simplicity let us disregard possible Bauschinger effects both in a crystal grain and in the aggregate as a whole, and suppose also that the critical shear stress is the same in all slip directions. If there were no work-hardening it is evident that the shape and size of the yield surface would be preserved during continued deformation, provided the distribution of orientations did not vary. (This last condition will not be satisfied whenever a preferred orientation is developed during cold-working). When work-hardening occurs the grains harden by different amounts, due partly to the non-uniform distortion and partly to their different orientations. However, to a first approximation, it seems reasonable to expect that the yield surface will simply increase in size without changing shape. The size will be proportional to some mean value of the critical shear stress.

There is some evidence (Taylor 1927) that the shear yield stress of a crystal is, very roughly, a function only of the total shear. That is,

$$\tau=F(\Sigma\gamma),$$

where the function F is the hardening curve in simple slip. Now we have seen that the work done on the aggregate unit cube during an increment of strain is

$$dW=\int(\tau\Sigma\,|d\gamma|)\,dV,$$
$$=\tau\Sigma\,|d\gamma|,$$

say, where τ and γ now denote mean values over the aggregate. Hence, approximately

$$dW=\tau\,d\tau/F'$$

where F′ is the slope of the shear-hardening curve. Since F′ is a function of τ only, so also is W. Conversely, the mean critical shear stress is a function only of the total plastic work.

These qualitative considerations lead, therefore, to the conclusion that the size of the yield surface is mainly a function of the total plastic work, and that other factors are secondary (in the absence of Bauschinger effects). This, indeed, is observed for face-centred cubic metals (see Hill 1950, pp. 27–32).

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